A FUNCTIONAL MODEL FOR THE TENSOR PRODUCT OF LEVEL 1 HIGHEST AND LEVEL -1 LOWEST MODULES FOR THE QUANTUM AFFINE ALGEBRA $U_q(\widehat{\mathfrak{sl}}_2)$

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Dedicated to Alain Lascoux on the occasion of his sixtieth birthday

Abstract. Let $V(\Lambda_i)$ (resp., $V(-\Lambda_j)$) be a fundamental integrable highest (resp., lowest) weight module of $U_q(\widehat{\mathfrak{sl}}_2)$. The tensor product $V(\Lambda_i)\otimes V(-\Lambda_j)$ is filtered by submodules $F_n=U_q(\widehat{\mathfrak{sl}}_2)(v_i\otimes\overline{v}_{n-i})$, $n\geq 0, n\equiv i-j \bmod 2$, where $v_i\in V(\Lambda_i)$ is the highest vector and $\overline{v}_{n-i}\in V(-\Lambda_j)$ is an extremal vector. We show that F_n/F_{n+2} is isomorphic to the level 0 extremal weight module $V(n(\Lambda_1-\Lambda_0))$. Using this we give a functional realization of the completion of $V(\Lambda_i)\otimes V(-\Lambda_j)$ by the filtration $(F_n)_{n\geq 0}$. The subspace of $V(\Lambda_i)\otimes V(-\Lambda_j)$ of $\mathfrak{sl}_2$-weight m is mapped to a certain space of sequences $(P_{n,l})_{n\geq 0, n\equiv i-j \bmod 2, n-2l=m}$, whose members $P_{n,l}=P_{n,l}(X_1,\ldots,X_l|z_1,\ldots,z_n)$ are symmetric polynomials in X_a and symmetric Laurent polynomials in z_k, with additional constraints. When the parameter q is specialized to $\sqrt{-1}$, this construction settles a conjecture which arose in the study of form factors in integrable field theory.

1. Introduction

For each fixed integer m and $i\in\{0,1\}$, let us consider sequences $\mathbf{p}=(P_{n,l})_{\substack{n\geq 0\\ n-2l=m}}$ of functions $P_{n,l}=P_{n,l}(X_1,\ldots,X_l|z_1,\ldots,z_n)$ satisfying the following conditions for all n,l:

(i) $P_{n,l}$ is a polynomial in $X_1,\ldots,X_l$ which is skew-symmetric when $l>1$,
(ii) $P_{n,l}$ is a symmetric Laurent polynomial in $z_1,\ldots,z_n$,
(iii) $\deg_{X_a} P_{n,l}\leq n-1$,
(iv)

$$P_{n+2,l+1}(X_1,\ldots,X_l,z^{-1}|z_1,\ldots,z_n,z,-z)\\ =z^{-n-1+i}\prod_{a=1}^{l}(1-X_a^2z^2)\cdot P_{n,l}(X_1,\ldots,X_l|z_1,\ldots,z_n).$$

Such sequences naturally arise in the form factor bootstrap approach to massive integrable models of quantum field theory [21]. Form factors are sequences of matrix elements of local fields taken between the vacuum and the asymptotic states. They are typically given by certain integrals involving polynomials $P_{n,l}$ of the type mentioned above, wherein $\alpha_a=-\log X_a$ are the integration variables and $\beta_j=\log z_j$ are the rapidity variables of asymptotic particles. More specifically, the sequences

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$\mathbf{p}$ satisfying (i)–(iv) appear in the sine-Gordon and the $SU(2)$ invariant Thirring models, and are called '∞-cycles' of weight m in [9]. The conditions (i)–(iii), along with (iv), which was originally proposed in [19] (see [9] in the present form with $i=0$), ensure the locality of the fields.

Denote by $\widehat{\mathcal{Z}}_{\mathbb{C}}^{\mathrm{skew}(i,j)}[m]$ the space of all ∞-cycles of weight m with $m \equiv i - j \bmod 2$. It was shown in [9] [1] that the space $\widehat{\mathcal{Z}}_{\mathbb{C}}^{\mathrm{skew}(i,j)} := \oplus_{\substack{m\in\mathbb{Z}\\ m\equiv i-j}} \widehat{\mathcal{Z}}_{\mathbb{C}}^{\mathrm{skew}(i,j)}[m]$ admits an action of the quantum affine algebra $U_{\sqrt{-1}}(\widehat{\mathfrak{sl}}_2)$ with the parameter $q=\sqrt{-1}$. It was conjectured further that $\widehat{\mathcal{Z}}_{\mathbb{C}}^{\mathrm{skew}(i,j)}$ is isomorphic to the tensor product module $V_{\sqrt{-1}}(\Lambda_i)\otimes V_{\sqrt{-1}}(-\Lambda_j)$ (with a proper completion, see below) of integrable modules of level 1 and level -1, respectively. The purpose of this paper is to clarify the representation theoretical origin of ∞-cycles, and to supply a proof of the above conjecture.

Though only the case $q=\sqrt{-1}$ is relevant to form factors, analogs of ∞-cycles exist also for generic q. In the below we outline their construction. Let $U_q = U_q(\widehat{\mathfrak{sl}}_2)$ be the quantum affine algebra over $K=\mathbb{C}(q)$, and let U'_q be the subalgebra with the scaling element being dropped. For $i,j=0,1$, let $V(\Lambda_i)$ (resp., $V(-\Lambda_j)$) be the integrable highest weight (resp., lowest weight) module with highest weight Λ_i (resp., $-\Lambda_j$) and highest weight vector v_i (resp., lowest weight vector $\bar v_{-j}$). Let further $\bar v_n \in V(-\Lambda_j)$ ($n\equiv j \bmod 2$) be an extremal vector obtained from $\bar v_{-j}\in V(-\Lambda_j)$ by the braid group action corresponding to the translation element $(s_0s_1)^{-(n+j)/2}$ of the Weyl group. In the tensor product $V(\Lambda_i)\otimes V(-\Lambda_j)$, the submodules

$$F_n^{(i,j)} = U_q(v_i\otimes \bar v_{n-i}) \qquad (n\ge 0, n\equiv i-j \bmod 2)$$

define a decreasing filtration

$$V(\Lambda_i)\otimes V(-\Lambda_j) = F^{(i,j)}_{|i-j|} \supset \cdots \supset F_n^{(i,j)} \supset F_{n+2}^{(i,j)} \supset \cdots . \tag{1.1}$$

Denote by $V_z = V\otimes K[z,z^{-1}]$ the evaluation module based on the two-dimensional space $V=Kv_+\oplus Kv_-$. Then there exists a U'_q-linear map

$$\psi_n \ : \ V(\Lambda_i)\otimes V(-\Lambda_j) \longrightarrow (V_{z_1}\otimes\cdots\otimes V_{z_n})^\wedge$$

such that $\psi_n(v_i\otimes \bar v_{n-i}) = v_+^{\otimes n}$ and $\psi_n(F^{(i,j)}_{n+2})=0$. Here the right hand side means the completion $\bigl(V_{z_1}\otimes\cdots\otimes V_{z_n}\bigr)\otimes_{K[z_1/z_2,\dots,z_{n-1}/z_n]} K[[z_1/z_2,\dots,z_{n-1}/z_n]]$. Furthermore, ψ_n induces an isomorphism

$$\phi_n \ : \ F_n^{(i,j)}/F_{n+2}^{(i,j)} \xrightarrow{\ \sim\ } V(n(\Lambda_1-\Lambda_0)) \tag{1.2}$$

between the associated graded space $F_n^{(i,j)}/F_{n+2}^{(i,j)}$ and the extremal weight module $V(n(\Lambda_1-\Lambda_0)) = U_q v_+^{\otimes n}$ of level 0 (Theorem 2.3).

For $0\le l\le n$, let $\mathcal{F}_{n,l}$ denote the space of symmetric polynomials $P(X_1,\dots,X_l)$, with coefficients in $K[z_1^{\pm1},\dots,z_n^{\pm1}]$, such that $\deg_{X_a} P\le n-1$ for each a and

$$P|_{X_1=q^{-2}X_2=z_j^{-1}} = 0 \text{ for each } j=1,\dots,n \quad \text{when } l>1.$$

[1] In [9], the space $\widehat{\mathcal{Z}}_{\mathbb{C}}^{\mathrm{skew}(0,j)}$ was denoted by $\widehat{\mathcal{Z}}^{(j)}$.

Let $\mathcal{F}_n = \oplus_{l=0}^n \mathcal{F}_{n,l}$. It is known [22] that there is an embedding of the tensor product of evaluation modules

$$\mathcal{C}_n \ : \ V_{z_1} \otimes \cdots \otimes V_{z_n} \longrightarrow \mathcal{F}_n,$$

which is $K[z_1, \ldots, z_n]$-linear. The subspace $(V_{z_1} \otimes \cdots \otimes V_{z_n})_m$ of $\mathfrak{sl}_2$-weight m is mapped to $\mathcal{F}_{n,l}$ where $n - 2l = m$. Let $\widehat{\mathcal{C}}_n : (V_{z_1} \otimes \cdots \otimes V_{z_n})^\wedge \to \widehat{\mathcal{F}}_n$ be the extension of $\mathcal{C}_n$ where $\widehat{\mathcal{F}}_n = \mathcal{F}_n \otimes_{K[z_1/z_2,\ldots,z_{n-1}/z_n]} K[[z_1/z_2, \ldots, z_{n-1}/z_n]]$. This is an isomorphism. Analogous isomorphisms hold also for modules over the integral form $U_A \subset U_q$ (where $A = \mathbb{C}[q, q^{-1}]$). It turns out that the image of $\varphi_n = \widehat{\mathcal{C}}_n \circ \psi_n$ is contained in the $\mathfrak{S}_n$-invariant subspace $\mathcal{F}_n^{\mathfrak{S}_n}$ of $\mathcal{F}_n$ (without completion). Here the symmetric group $\mathfrak{S}_n$ acts on $\mathcal{F}_n$ by the permutation of the variables $z_1 \ldots, z_n$. The image of the map

$$\varphi = \prod_n \varphi_n \ : \ V(\Lambda_i) \otimes V(-\Lambda_j) \longrightarrow \prod_{\substack{n \geq 0 \\ n \equiv i-j \bmod 2}} \mathcal{F}_n^{\mathfrak{S}_n}$$

is contained in the subspace defined as follows. Denote by $\widehat{\mathcal{Z}}^{(i,j)}[m]$ the space of all sequences $\mathbf{p} = (P_{n,l})_{\substack{n \geq 0 \\ n-2l=m}}$ of polynomials $P_{n,l} \in \mathcal{F}_{n,l}^{\mathfrak{S}_n}$, satisfying the property:

$$\text{(iv)}' \qquad \begin{aligned} &P_{n+2,l+1}(X_1, \ldots, X_l, z^{-1} | z_1, \ldots, z_n, z, q^2 z) \\ &= z^{-n-1+i} \textstyle\prod_{a=1}^l (1 - q^{-2} X_a z)(1 - q^2 X_a z) \cdot P_{n,l}(X_1, \ldots, X_l | z_1, \ldots, z_n). \end{aligned}$$

We set $\widehat{\mathcal{Z}}^{(i,j)} = \oplus_{m \in \mathbb{Z}} \widehat{\mathcal{Z}}^{(i,j)}[m] \subset \prod_{\substack{n \geq 0 \\ n \equiv i-j \bmod 2}} \mathcal{F}_n^{\mathfrak{S}_n}$. The image of $V(\Lambda_i) \otimes V(-\Lambda_j)$ is contained in this subspace, and moreover, the completion of $V(\Lambda_i) \otimes V(-\Lambda_j)$ by the filtration $\{F_n^{(i,j)}\}$ is isomorphic to $\widehat{\mathcal{Z}}^{(i,j)}$ (Theorem 3.7).

By a simple transformation, the specialization of (the integral form of) $\widehat{\mathcal{Z}}^{(i,j)}$ to $q = \sqrt{-1}$ is mapped injectively to the space of ∞-cycles $\widehat{\mathcal{Z}}_{\mathbb{C}}^{\mathrm{skew}(i,j)}$. From this follows the conjectured isomorphism in the original setting.

Quite generally, it is known [1] for an arbitrary quantized affine algebra that the tensor product of highest and lowest modules with total level zero admits a filtration with a property similar to (1.2). However each filter is in general not generated by tensor products of extremal vectors. It would be interesting to study their structure. In particular the filtration on the tensor product induces a filtration on the lowest weight module. In the case of $\widehat{\mathfrak{sl}}_2$ we give a conjecture on the character of the associated graded space for the latter (see (3.25)).

The text is organized as follows. In Section 2, we give a brief review on extremal weight modules and set up the notation. We then introduce the filtration (1.1) of the tensor product of level 1 highest and level -1 lowest modules, and prove the isomorphism (1.2). In Section 3, we discuss the polynomial realization of the tensor product $V(\Lambda_i) \otimes V(-\Lambda_j)$ and the associated graded spaces of the filtration $\{F_n^{(i,j)}\}$. The main results are stated in Theorem 3.7 and Theorem 3.9.

For the reader's convenience, we summarize in Appendix some basic facts concerning crystal and global basis of extremal weight modules used in the text. We

also give a brief account of the filtration of the tensor product modules $V(\xi)\otimes V(-\eta)$ for general quantum affine algebras.

2. Filtration by extremal vectors

2.1. **Notation.** First we fix our notation concerning quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$. Set $I=\{0,1\}$. Let $P=\mathbb{Z}\Lambda_0\oplus\mathbb{Z}\Lambda_1\oplus\mathbb{Z}\delta$ be the weight lattice for $\widehat{\mathfrak{sl}}_2$, $P^*=\mathbb{Z}h_0\oplus\mathbb{Z}h_1\oplus\mathbb{Z}d$ its dual lattice and $P_+=\{\lambda\in P|\langle\lambda,h_i\rangle\geq 0\,(i\in I)\}$. The quantum affine algebra $U_q=U_q(\widehat{\mathfrak{sl}}_2)$ is the algebra over $K=\mathbb{C}(q)$ generated by e_i,f_i $(i\in I)$ and q^h $(h\in P^*)$, under the defining relations

$$
\begin{aligned}
&q^hq^{h'}=q^{h+h'},\quad q^0=1,\\
&q^he_iq^{-h}=q^{\langle h,\alpha_i\rangle}e_i,\quad q^hf_iq^{-h}=q^{-\langle h,\alpha_i\rangle}f_i,\\
&[e_i,f_j]=\delta_{ij}\frac{t_i-t_i^{-1}}{q-q^{-1}},\\
&\sum_{r=0}^{3}(-1)^re_i^{(3-r)}e_je_i^{(r)}=0\quad(i\neq j),\\
&\sum_{r=0}^{3}(-1)^rf_i^{(3-r)}f_jf_i^{(r)}=0\quad(i\neq j).
\end{aligned}
$$

Here $t_i=q^{h_i}$, $\alpha_1=2(\Lambda_1-\Lambda_0)$, $\alpha_0=\delta-\alpha_1$, and for an element $x\in U_q$, we denote by $x^{(r)}$ the divided power $x^r/[r]!$, where $[r]!=\prod_{j=1}^r[j]$, $[j]=(q^j-q^{-j})/(q-q^{-1})$. The element $C:=t_0t_1$ is central, and $D:=q^d$ is the scaling element. We will use the coproduct

$$(2.1)\ \Delta(e_i)=e_i\otimes t_i^{-1}+1\otimes e_i,\ \Delta(f_i)=f_i\otimes 1+t_i\otimes f_i,\ \Delta(q^h)=q^h\otimes q^h.$$

We denote by W the Weyl group for $\widehat{\mathfrak{sl}}_2$.

We say that a U_q-module M is integrable if the action of e_i,f_i $(i\in I)$ is locally nilpotent and $M=\oplus_{\mu\in P}M_\mu$, $M_\mu:=\{u\in M\mid q^hu=q^{\langle h,\mu\rangle}u\ (h\in P^*)\}$. For $u\in M_\mu$ we write $\mathrm{wt}\,u=\mu$, $Du=q^{\deg u}u$.

We will consider the specialization at $q=\sqrt{-1}$. For this purpose, we need the integral form of U_q. We set $A=\mathbb{C}[q,q^{-1}]$. Let U_A be the A-subalgebra of U_q generated by $e_i^{(r)},f_i^{(r)}$ $(i\in I,\ r\in\mathbb{Z}_{\geq 0})$ and q^h $(h\in P^*)$. For $\epsilon\in\mathbb{C}\backslash\{0\}$, the specialization U_ϵ is the $\mathbb{C}$-algebra $U_A/U_A(q-\epsilon)$. We denote by U_q^+ (resp., U_q^-) the subalgebra of U_q generated by e_i $(i\in I)$ (resp., f_i $(i\in I)$).

We denote by $U_q^{\geq 0}$ the subalgebra of U_q generated by e_i, q^h $(i\in I,h\in P^*)$ and f_1, and by $U_A^{\geq 0}$ the A-subalgebra generated by $e_i^{(r)}$, q^h and $f_1^{(r)}$ for $i\in I,h\in P^*$, $r\in\mathbb{Z}_{\geq 0}$. Likewise, we define $U_q^{\leq 0}$ and $U_A^{\leq 0}$ by changing e_i (or $e_i^{(r)}$) to f_i (or $f_i^{(r)}$) and f_1 (or $f_1^{(r)}$) to e_1 (or $e_1^{(r)}$) in the definition above.

Let M be a K-vector space. An A-submodule M_A of M is called an A-lattice of M if it is a free A-module and $M=M_A\otimes_A K$. For an A-lattice M_A and $\epsilon\in\mathbb{C}\backslash\{0\}$, we write $(M_A)_\epsilon=M_A/(q-\epsilon)M_A$. We call it the specialization of M_A at $q=\epsilon$. If there

is no fear of confusion, we abbreviate $(M_A)_\epsilon$ to M_ϵ. When we specialize a U_q-module M to $q=\epsilon$, we must first choose an A-lattice of M which is stable under the action of U_A. Then, the specialization M_ϵ admits a U_ϵ-action and we obtain a U_ϵ-module. We note that different A-lattices may lead to non-isomorphic U_ϵ-modules.

Let M be an integrable U_q-module. Consider the bi-grading of $M=\oplus_{a,n\in\mathbb{Z}}M_{a,n}$, where $M_{a,n}=\{u\in M \mid Du=q^a u, t_1u=q^n u\}$. We define its character by

$$\mathrm{ch}_{v,z}M := \sum_{a,n\in\mathbb{Z}} \dim_K M_{a,n} v^a z^n. \tag{2.2}$$

If M is a bi-graded $\mathbb{C}$-vector space (resp., a bi-graded free A-module) we define its character by (2.2) replacing the dimension over K by the dimension over $\mathbb{C}$ (resp., the rank over A). If M_A is an A-lattice of M, the characters $\mathrm{ch}_{v,z}M$, $\mathrm{ch}_{v,z}M_A$, $\mathrm{ch}_{v,z}M_\epsilon$ are all equal. We note that the character is well-defined only if each subspace $M_{a,n}$ is finite-dimensional (or of finite rank). In fact, U_q-modules we consider in this paper do not necessarily satisfy this property, e.g., $V(2(\Lambda_1-\Lambda_0))$ given in the next section.

2.2. **Extremal weight modules.** We recall the notion of extremal weight modules over U_q, introduced in [11] for general quantized enveloping algebras. Let $\lambda\in P$. In the present case of affine type algebras, the extremal weight module $V(\lambda)$ is characterized as the universal integrable U_q-module with the following defining relations:

$$V(\lambda)=U_q u_\lambda \text{ where } \mathrm{wt}\, u_\lambda=\lambda, \tag{2.3}$$

$$\mathrm{wt}\, V(\lambda)\subset \text{the convex hull of } W\lambda. \tag{2.4}$$

We define an A-lattice of $V(\lambda)$ by $V_A(\lambda)=U_A u_\lambda$. We denote its specialization to $q=\epsilon\in\mathbb{C}\backslash\{0\}$ by $V_\epsilon(\lambda)$.

The extremal weight module $V(\lambda)$ has a family of extremal vectors $S_w u_\lambda$ indexed by $w\in W$. They are defined by $S_{\mathrm{id}}u_\lambda=u_\lambda$ and

$$S_{s_i w}u_\lambda=\begin{cases} f_i^{(\langle h_i,w\lambda\rangle)}S_w u_\lambda & \text{if } \langle h_i,w\lambda\rangle\geq 0,\\ e_i^{(-\langle h_i,w\lambda\rangle)}S_w u_\lambda & \text{if } \langle h_i,w\lambda\rangle\leq 0.\end{cases} \tag{2.5}$$

For $w\in W$, we have $V(\lambda)=U_qS_wu_\lambda$. There is a canonical isomorphism

$$V(w\lambda)=U_q u_{w\lambda}\xrightarrow{\sim}V(\lambda) \tag{2.6}$$

sending $u_{w\lambda}$ to $S_w u_\lambda$.

If $\lambda\in P_+$, $V(\lambda)=U_q u_\lambda$ is the integrable highest weight module with highest weight λ and highest weight vector u_λ. Similarly, $V(-\lambda)=U_q u_{-\lambda}$ is the integrable lowest weight module with lowest weight $-\lambda$ and lowest weight vector $u_{-\lambda}$. The following result [15] is basic in our study.

Proposition 2.1. *Let $\lambda,\mu\in P_+$ and $\epsilon\in\mathbb{C}\backslash\{0\}$. The tensor products $V(\lambda)\otimes V(-\mu)$, $V_A(\lambda)\otimes V_A(-\mu)$, $V_\epsilon(\lambda)\otimes V_\epsilon(-\mu)$ have the cyclic vector $u_\lambda\otimes u_{-\mu}$. They are characterized as the universal cyclic module with the cyclic vector v, with weight*

condition $\mathrm{wt}\, v = \lambda - \mu$ *and the defining relations*

$$f_i^{(r)} v = 0 \text{ for any } r \geq \langle h_i, \lambda \rangle + 1, \tag{2.7}$$

$$e_i^{(r)} v = 0 \text{ for any } r \geq \langle h_i, \mu \rangle + 1. \tag{2.8}$$

In this paper, we specifically consider the U_q-modules $V(\Lambda_i)$ and $V(-\Lambda_i)$ where $i = 0, 1$. Let $v_i := u_{\Lambda_i} \in V(\Lambda_i)$ be the highest weight vector, and $\overline{v}_{-i} := u_{-\Lambda_i} \in V(-\Lambda_i)$ the lowest weight vector. For $n \in \mathbb{Z}$ such that $n \equiv i \bmod 2$, consider the extremal vectors

$$v_n := S_{(s_0 s_1)^{(n-i)/2}} v_i, \quad \mathrm{wt}\, v_n = \Lambda_0 + n(\Lambda_1 - \Lambda_0) - \frac{n^2 - i}{4}\delta, \tag{2.9}$$

$$\overline{v}_n := S_{(s_0 s_1)^{(-n-i)/2}} \overline{v}_{-i}, \quad \mathrm{wt}\, \overline{v}_n = -\Lambda_0 + n(\Lambda_1 - \Lambda_0) + \frac{n^2 - i}{4}\delta. \tag{2.10}$$

We shall deal also with level 0 weights $\lambda = n(\Lambda_1 - \Lambda_0) + r\delta$ $(n, r \in \mathbb{Z})$. Let U'_q be the K-subalgebra of U_q generated by $e_i, f_i, t_i^{\pm 1}$ $(i \in I)$. We have an isomorphism $V(\lambda + m\delta) \simeq V(\lambda)$ as U'_q-modules for any $m \in \mathbb{Z}$. Indeed, as U'_q-modules an isomorphism is defined by sending $u_{\lambda + m\delta}$ to u_λ. If $m \neq 0$, this map is not D-linear. The degree d in $V(\lambda + m\delta)$ and that in $V(\lambda)$ differs by m. From the structure of the modules, which we will describe below, it is easy to see that there exists an isomorphism as U_q-modules between $V(\lambda + m\delta)$ and $V(\lambda)$ if and only if $m \equiv 0 \bmod n$.

Set $V = Kv_+ \oplus Kv_-$, $V_z = V \otimes K[z^{\pm 1}]$. We abbreviate $v_\pm \otimes z^m$ to $z^m v_\pm$ and set $Dz^m v_\pm = q^m z^m v_\pm$. Define further an action of U'_q on V_z by the rule that it commutes with the multiplication by z, and

$$\begin{array}{lllll} t_0 v_\pm = q^{\mp 1} v_\pm, & e_0 v_+ = z v_-, & e_0 v_- = 0, & f_0 v_+ = 0, & f_0 v_- = z^{-1} v_+, \\ t_1 v_\pm = q^{\pm 1} v_\pm, & e_1 v_+ = 0, & e_1 v_- = v_+, & f_1 v_+ = v_-, & f_1 v_- = 0. \end{array} \tag{2.11}$$

Then we have an isomorphism of U_q-modules $V_z \simeq V(\Lambda_1 - \Lambda_0)$ which sends v_+ to $u_{\Lambda_1 - \Lambda_0}$. We have also $V_{A,z} \simeq V_A(\Lambda_1 - \Lambda_0)$, where $V_{A,z} = (Av_+ \oplus Av_-) \otimes_A A[z^{\pm 1}]$. In addition, $S_{(s_0 s_1)^m} v_+ = z^{-m} v_+$ and $S_{s_1 (s_0 s_1)^m} v_+ = z^{-m} v_-$.

For general $n \in \mathbb{Z}_{\geq 0}$, we identify $V_z^{\otimes n}$ with the vector space $V^{\otimes n} \otimes K[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$. The U_q module $V(n(\Lambda_1 - \Lambda_0))$ is isomorphic to a submodule $U_q v_+^{\otimes n}$ of $V_z^{\otimes n}$. We identify them also, by the identification $v_+^{\otimes n} = u_{n(\Lambda_1 - \Lambda_0)}$. Denoting by $\mathfrak{S}_n$ the symmetric group on n letters, we have

$$\oplus_{m \in \mathbb{Z}} V(n(\Lambda_1 - \Lambda_0))_{n(\Lambda_1 - \Lambda_0) + m\delta} = Kv_+^{\otimes n} \otimes K[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]^{\mathfrak{S}_n}. \tag{2.12}$$

Viewed as a subspace of $V^{\otimes n} \otimes K[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$, $V(n(\Lambda_1 - \Lambda_0))$ is invariant under multiplication by symmetric Laurent polynomials, and the multiplication commutes with the U'_q-action. We have $S_{(s_0 s_1)^m} v_+^{\otimes n} = z^{-m} v_+^{\otimes n}$ and $S_{s_1 (s_0 s_1)^m} v_+^{\otimes n} = z^{-m} v_-^{\otimes n}$ where $z = z_1 \cdots z_n$.

The U_q-module $V(\lambda)$ with $\lambda = n(\Lambda_1 - \Lambda_0)$ is also characterized as the universal integrable module $U_q u_\lambda$ satisfying the following properties.

$$\mathrm{wt}\, u_\lambda = \lambda, \tag{2.13}$$

$$V(\lambda)_\xi = 0 \text{ if } \xi \in \lambda + \mathbb{Z}_{>0}\alpha_1 + \mathbb{Z}\delta. \tag{2.14}$$

2.3. **Filtration of $V(\Lambda_i)$.** We introduce a decreasing filtration of $V(\Lambda_i)$ by $U_q^{\le 0}$-modules. Namely, we set

$$F_n^{(i)} := U_q^{\le 0} v_n \quad (n \ge 0, n \equiv i \bmod 2).$$

Then, we have $F_n^{(i)} \supset F_{n+2}^{(i)}$ and $\cap_n F_n^{(i)} = 0$. Similarly, we define the filtration of $V(-\Lambda_i)$ by $U_q^{\ge 0}$-modules.

$$\overline{F}_n^{(i)} := U_q^{\ge 0} \overline{v}_n \quad (n \ge 0, n \equiv i \bmod 2).$$

2.4. **Filtration of $V(\Lambda_i) \otimes V(-\Lambda_j)$.** Hereafter we fix $i, j \in I$ and consider the tensor product $V(\Lambda_i) \otimes V(-\Lambda_j)$. For $n \ge 0$ such that $n \equiv i - j \bmod 2$, we set

$$F_n^{(i,j)} := U_q(v_i \otimes \overline{v}_{n-i}).$$

Note that $F_n^{(i,j)} = U_q(v_{-n+j} \otimes \overline{v}_{-j})$. This defines a decreasing filtration by U_q-submodules

$$V(\Lambda_i) \otimes V(-\Lambda_j) = F_{|i-j|}^{(i,j)} \supset F_{|i-j|+2}^{(i,j)} \supset \cdots .$$

Similarly, we set $F_{A,n}^{(i,j)} := U_A(v_i \otimes \overline{v}_{n-i})$, $F_{\epsilon,n}^{(i,j)} := U_\epsilon(v_i \otimes \overline{v}_{n-i})$.

Proposition 2.2.

$$\bigcap_{\substack{n \ge 0 \\ n \equiv i-j \bmod 2}} F_n^{(i,j)} = 0.$$

A proof is given in Appendix.

Now we state one of the main results in this paper.

Theorem 2.3. *We have an isomorphism of U_q'-modules*

$$\phi_n : F_n^{(i,j)}/F_{n+2}^{(i,j)} \simeq V(n(\Lambda_1 - \Lambda_0)) \quad (n \ge 0, n \equiv i - j \bmod 2),$$

which sends $v_i \otimes \overline{v}_{n-i}$ to $v_+^{\otimes n}$. For a weight vector $u \in F_n^{(i,j)}/F_{n+2}^{(i,j)}$, we have

$$\deg u = \deg \phi_n(u) + \frac{(n-i)^2 - j}{4}.$$

The proof is based on the following two propositions.

Proposition 2.4. *We have*

$$(F_n^{(i,j)}/F_{n+2}^{(i,j)})_\xi = 0 \quad \text{for} \quad \xi \in n(\Lambda_1 - \Lambda_0) + \mathbb{Z}_{>0}\alpha_1 + \mathbb{Z}\delta.$$

Proposition 2.5. *There exists a U_q'-linear surjection*

$$\tilde{\psi}_n : F_n^{(i,j)} \longrightarrow V(n(\Lambda_1 - \Lambda_0))$$

which sends $v_i \otimes \overline{v}_{n-i}$ to $v_+^{\otimes n}$.

We show Proposition 2.4 in Sections 2.5–2.6, and Proposition 2.5 in Section 2.7. Assuming them, let us prove Theorem 2.3.

Proof of Theorem 2.3. Set $\lambda = n(\Lambda_1 - \Lambda_0)$. The module $F_n^{(i,j)}/F_{n+2}^{(i,j)}$ is generated by a vector $u := v_i \otimes \overline{v}_{n-i}$ of weight $\lambda + ((n-i)^2 - j)/4 \cdot \delta$, and the weight spaces are restricted as in Proposition 2.4. From the characterization (2.13), (2.14) of extremal weight modules $V(\lambda)$, we have a surjection of U_q'-modules

$$V(\lambda) \longrightarrow F_n^{(i,j)}/F_{n+2}^{(i,j)}. \tag{2.15}$$

Since $\mathrm{wt}\,\psi_n(v_i \otimes \overline{v}_{n+2-i}) + \mathbb{Z}\delta$ does not appear in the weights of $V(\lambda)$, we have $\tilde{\psi}_n(F_{n+2}^{(i,j)}) = 0$. Hence we have also a surjection

$$F_n^{(i,j)}/F_{n+2}^{(i,j)} \longrightarrow V(\lambda). \tag{2.16}$$

The mappings (2.15) and (2.16) exchange the cyclic vector $u_\lambda \in V(\lambda)$ and the cyclic vector $u \in F_n^{(i,j)}/F_{n+2}^{(i,j)}$. These maps are therefore U_q'-isomorphisms. The statement about the degree follows from (2.9) and (2.10). □

Proposition 2.6. *The isomorphism ϕ_n in Theorem 2.3 induces the following isomorphisms*:

$$F_{A,n}^{(i,j)}/F_{A,n+2}^{(i,j)} \simeq V_A(n(\Lambda_1 - \Lambda_0)),$$
$$F_{\epsilon,n}^{(i,j)}/F_{\epsilon,n+2}^{(i,j)} \simeq V_\epsilon(n(\Lambda_1 - \Lambda_0)).$$

A proof is given in Appendix.

The following result is a consequence of Theorem 2.3. We use a result in [14] in the proof.

Proposition 2.7.

$$\overline{F}_n^{(j)}/\overline{F}_{n+2}^{(j)} \simeq U_q^{\geq 0} v_+^{\otimes n}. \tag{2.17}$$

Proof. we show that

$$(v_0 \otimes \overline{F}_n^{(j)}) \cap U_q(v_0 \otimes \overline{v}_{-(n+2)}) = v_0 \otimes \overline{F}_{n+2}^{(j)}.$$

The inclusion $\supset$ is clear. In order to show the other inclusion, we apply Corollary 3.2 in [14] by setting $\lambda = \Lambda_0$, $\mu = \mathrm{wt}\,\overline{v}_{-(n+2)}$. Set $M = \overline{F}_n^{(j)} = U_q^+ \overline{v}_{-n}$ and $N = \overline{F}_{n+2}^{(j)} = U_q^+ \overline{v}_{-(n+2)}$. Note that $v_0 = u_\lambda$ and $V(\mu) \simeq V(-\Lambda_j)$. We have

$$(u_\lambda \otimes M) \cap U_q(u_\lambda \otimes \overline{v}_{-(n+2)}) \subset (u_\lambda \otimes V(\mu)) \cap U_q(u_\lambda \otimes \overline{v}_{-(n+2)}) = u_\lambda \otimes N.$$

Noting that $U_q(v_0 \otimes \overline{v}_{n+2}) = U_q(v_0 \otimes \overline{v}_{-(n+2)})$, we have

$$M/N \simeq v_0 \otimes M/((v_0 \otimes M) \cap U_q(v_0 \otimes \overline{v}_{n+2})) \subset F_n^{(0,j)}/F_{n+2}^{(0,j)} \simeq V(n(\Lambda_1 - \Lambda_0)).$$

Since $v_0 \otimes \overline{v}_n$ is identified with $v_+^{\otimes n}$ in $V(n(\Lambda_1 - \Lambda_0)) \subset V_z^{\otimes n}$, we have (2.17). □

See Corollary 3.10 for the character of (2.17). From the proof of Proposition 2.7 we have also

$$F_n^{(0,j)} \cap (v_0 \otimes V(-\Lambda_j)) = v_0 \otimes \overline{F}_n^{(j)}. \tag{2.18}$$

2.5. **Vertex operator realization.** To show Proposition 2.4, we utilize the realization of the modules $V(\Lambda_i)$, $V(-\Lambda_j)$ in terms of vertex operators. For that purpose, consider the Drinfeld generators of U_q, $x^\pm_k$ ($k \in \mathbb{Z}$), b_n ($n \in \mathbb{Z}\backslash\{0\}$) and $t_1^{\pm 1}$, $C^{\pm 1}$, $D^{\pm 1}$ obeying the following relations [2].

$$Db_nD^{-1} = q^n b_n,\ \ Dx^\pm_n D^{-1} = q^n x^\pm_n, \tag{2.19}$$

$$t_1 b_n = b_n t_1, \quad t_1 x^\pm_n t_1^{-1} = q^{\pm 2} x^\pm_n, \tag{2.20}$$

$$[b_m, b_n] = m\frac{[2m]}{[m]^2}\frac{C^m - C^{-m}}{q-q^{-1}}\delta_{m+n,0}, \tag{2.21}$$

$$[b_n, x^\pm_k] = \pm\frac{[2n]}{[n]}C^{(n\mp|n|)/2}x^\pm_{k+n}, \tag{2.22}$$

$$x^\pm_{k+1}x^\pm_l - q^{\pm 2}x^\pm_l x^\pm_{k+1} = q^{\pm 2}x^\pm_k x^\pm_{l+1} - x^\pm_{l+1}x^\pm_k, \tag{2.23}$$

$$[x^+_k, x^-_l] = \frac{C^{-l}\varphi^+_{k+l} - C^{-k}\varphi^-_{k+l}}{q-q^{-1}}. \tag{2.24}$$

Here

$$\sum_{k\in\mathbb{Z}} \varphi^\pm_{\pm k} z^k = t_1^{\pm 1}\exp\left(\pm\sum_{n=1}^\infty \frac{(q^n - q^{-n})}{n} b_{\pm n} z^n\right).$$

They are related to the Chevalley generators by [3]

$$e_1 t_1 = x^+_0,\ t_1^{-1} f_1 = x^-_0,\ e_0 C = x^-_1,\ C^{-1} f_0 = x^+_{-1},\ t_1 = t_1,\ t_0 = C t_1^{-1}.$$

The subalgebra U_A contains $(x^\pm_n)^{(r)}$ and b_n. We shall work with the generating series

$$x^\pm(z) = \sum_{n\in\mathbb{Z}} x^\pm_n z^{-n-1}.$$

Set

$$V'(\Lambda_i) = K[b_n \mid n \in \mathbb{Z}_{<0}] \otimes \left(\oplus_{m\in\mathbb{Z}} K e^{\Lambda_i + m\alpha_1}\right),$$
$$V'_A(\Lambda_i) = A[b_n \mid n \in \mathbb{Z}_{<0}] \otimes \left(\oplus_{m\in\mathbb{Z}} A e^{\Lambda_i + m\alpha_1}\right).$$

Let C, b_n ($n \in \mathbb{Z}\setminus\{0\}$) and ∂ act on an element $P\otimes e^\beta \in V'(\Lambda_i)$ as

$$C(P\otimes e^\beta) = q(P\otimes e^\beta),$$
$$b_n(P\otimes e^\beta) = \begin{cases}(b_n P)\otimes e^\beta & (n<0),\\ [b_n, P]\otimes e^\beta & (n>0),\end{cases}$$
$$\partial(P\otimes e^\beta) = \langle h_1, \beta\rangle P\otimes e^\beta.$$

We introduce the grading on $V'(\Lambda_i)$ by setting

$$\deg b_n = n, \quad \deg e^{\Lambda_i + m\alpha_1} = -m^2 - im.$$

[2]These generators are the same as those in [7], except $C = \gamma$, $D = q^d$ and $b_n = (n/[n])\gamma^{n/2}a_n$.

[3]To conform with the coproduct (2.1), we have changed the identification slightly from [7]. With this identification, the formulas of coproduct for the Drinfeld generators are unchanged.

Define the action of U_q on $V'(\Lambda_i)$ by

$$x^+(z)\cdot u=\exp\Bigl(-\sum_{n<0}\frac{b_n}{n}z^{-n}\Bigr)\exp\Bigl(-\sum_{n>0}\frac{b_n}{n}(qz)^{-n}\Bigr)\,e^{\alpha_1}z^{\partial}\,u, \tag{2.25}$$

$$x^-(z)\cdot u=\exp\Bigl(\sum_{n<0}\frac{b_n}{n}(qz)^{-n}\Bigr)\exp\Bigl(\sum_{n>0}\frac{b_n}{n}z^{-n}\Bigr)\,e^{-\alpha_1}z^{-\partial}\,u, \tag{2.26}$$

$$t_1\cdot u=q^{\partial}u,\ \ D\cdot u=q^{\deg u}\,u, \tag{2.27}$$

where $u\in V'(\Lambda_i)$ is assumed to be homogeneous.

Proposition 2.8. [5, 4] *For $i=0,1$, $V'(\Lambda_i)$ is a U_q-module isomorphic to $V(\Lambda_i)$, and $V'_A(\Lambda_i)$ is isomorphic to its A-form $V_A(\Lambda_i)$.*

We will need also the realization of level -1 modules. Set

$$\begin{aligned}
&V'(-\Lambda_i)=K[b_n\mid n\in\mathbb{Z}_{>0}]\otimes\bigl(\oplus_{m\in\mathbb{Z}}Ke^{-\Lambda_i-m\alpha_1}\bigr),\\
&V'_A(-\Lambda_i)=A[b_n\mid n\in\mathbb{Z}_{>0}]\otimes\bigl(\oplus_{m\in\mathbb{Z}}Ae^{-\Lambda_i-m\alpha_1}\bigr),\\
&C(P\otimes e^{\beta})=q^{-1}(P\otimes e^{\beta}),\\
&b_n(P\otimes e^{\beta})=\begin{cases}(b_nP)\otimes e^{\beta} & (n>0),\\ [b_n,P]\otimes e^{\beta} & (n<0),\end{cases},\\
&\partial(P\otimes e^{\beta})=\langle h_1,\beta\rangle P\otimes e^{\beta},
\end{aligned}$$

and

$$\deg b_n=n,\quad \deg e^{-\Lambda_i-m\alpha_1}=m^2+im.$$

Let U_q act on $u\in V'(-\Lambda_i)$ as

$$x^+(z).u=\exp\Bigl(\sum_{n>0}\frac{b_n}{n}(q^{-1}z)^{-n}\Bigr)\exp\Bigl(\sum_{n<0}\frac{b_n}{n}z^{-n}\Bigr)\,z^{-\partial}e^{\alpha_1}\,u, \tag{2.28}$$

$$x^-(z).u=\exp\Bigl(-\sum_{n>0}\frac{b_n}{n}z^{-n}\Bigr)\exp\Bigl(-\sum_{n<0}\frac{b_n}{n}(q^{-1}z)^{-n}\Bigr)\,z^{\partial}e^{-\alpha_1}\,u, \tag{2.29}$$

$$t_1.u=q^{\partial}u,\ \ D.u=q^{\deg u}\,u. \tag{2.30}$$

Proposition 2.9. *For $j=0,1$, $V'(-\Lambda_j)$ is a U_q-module isomorphic to $V(-\Lambda_j)$, and $V'_A(-\Lambda_j)$ is isomorphic to its A-form $V_A(-\Lambda_j)$.*

For $i=0,1$ and $m\in\mathbb{Z}$ with $m\equiv i \bmod 2$, we set

$$v'_m=e^{\Lambda_i+(m-i)\alpha_1/2},\quad \overline{v}'_m=e^{-\Lambda_i+(m+i)\alpha_1/2}.$$

The following lemma shows that, up to a non-zero scalar multiple, they are extremal vectors in $V(\Lambda_i)$ and $V(-\Lambda_i)$, respectively.

Lemma 2.10. *The following relations hold for $m \geq 0$.*

$$\begin{aligned}
(x_0^-)^{(m)} v'_m &= (-q)^{m(m-1)/2} v'_{-m},\\
(x_0^-)^{(m)} \overline{v}'_m &= (-q)^{-m(m-1)/2} \overline{v}'_{-m},\\
(x_{-1}^+)^{(m+1)} v'_{-m} &= (-q)^{-m(m+1)/2} v'_{m+2},\\
(x_{-1}^+)^{(m+1)} \overline{v}'_{-m-2} &= (-q)^{m(m+1)/2} \overline{v}'_{m}.
\end{aligned}$$

Proof. From (2.26), it is straightforward to verify that

$$\begin{aligned}
&x^-(z_1)\dots x^-(z_m) v'_m\\
&= \prod_{j=1}^m z_j^{-m} \prod_{1\le j<k\le m} (z_j - z_k)(z_j - q^2 z_k) \times \exp\Bigl(-\sum_{n>0} \frac{b_{-n}}{n} \sum_{j=1}^m (q z_j)^n\Bigr)\, v'_{-m}.
\end{aligned}$$

Hence $(x_0^-)^m v'_m$ is the coefficient of $(z_1 \dots z_m)^0$ in

$$\prod_{j=1}^m z_j^{-m+1} \prod_{1\le j<k\le m} (z_j - z_k)(z_j - q^2 z_k) \times v'_{-m}.$$

We may replace this expression by its symmetrization with respect to $z_1, \dots, z_m$. Using

$$\sum_{\sigma\in S_m} \prod_{j<k} \frac{z_{\sigma(j)} - q^2 z_{\sigma(k)}}{z_{\sigma(j)} - z_{\sigma(k)}} = q^{m(m-1)/2}[m]!,$$

we find the first formula of Lemma. The other ones are obtained similarly. □

2.6. **Proof of Proposition 2.4.** Proposition 2.4 will follow if we show the following formula:

Proposition 2.11.

$$\begin{aligned}
(2.31)\qquad &\Bigl(\sum_{n\ge 0} x_n^+ z^{-n-1}\Bigr)(v'_i \otimes \overline{v}'_m)\\
&\quad = q^{2m+i+2} z^{-m-2}\, \exp\bigl(\sum_{k>0} \frac{b_k}{k} (q^{-1} z)^{-k}\bigr)\, (v'_i \otimes \overline{v}'_{m+2}),\\
(2.32)\qquad &\Bigl(\sum_{n<0} x_n^+ z^{-n-1}\Bigr)(v'_i \otimes \overline{v}'_m)\\
&\quad = (-q^2)^{m+i+1} z^i\, \exp\bigl(\sum_{k>0} \frac{b_{-k}}{k} z^k\bigr) (x_{-1}^+)^{(m+2+i)} (x_0^-)^{(m+2+i)}\, (v'_i \otimes \overline{v}'_{m+2}).
\end{aligned}$$

Proof. For the calculation, we use the following formulas proved in [2]. Let $U_q^{+,0}$ (resp., $U_q^{-,0}$) be the subalgebra generated by e_i and q^h (resp., f_i and q^h) with $i \in I$, $h \in P^*$. Denote by $N_{\ge 0}^+$ (resp., $N_{>0}^-$, $N_{<0}^+$, $N_{\le 0}^-$) the linear span of the elements x_n^+

$(n\ge 0)$ (resp., x^-_n $(n>0)$, x^+_n $(n<0)$, x^-_n $(n\le 0)$). Then

$$
\begin{aligned}
\Delta x^+_n &\equiv x^+_n\otimes C^n+\sum_{j=0}^{n}C^{2j}\varphi^+_{n-j}\otimes C^{n-j}x^+_j\\
&\qquad\qquad \bmod\ U_q^{+,0}N^-_{>0}\otimes U_q^{+,0}(N^+_{\ge 0})^2\quad (n\ge 0),\\
\Delta x^+_{-n} &\equiv x^+_{-n}\otimes C^{-2n+1}+\sum_{j=0}^{n-1}C^{2j-n}\varphi^-_{-j}\otimes C^{-2j}x^+_{-n+j}\\
&\qquad\qquad \bmod\ U_q^{-,0}N^-_{\le 0}\otimes U_q^{-,0}(N^+_{<0})^2\quad (n>0),\\
\Delta x^-_n &\equiv C^{2n-1}\otimes x^-_n+\sum_{j=0}^{n-1}C^{2j}x^-_{n-j}\otimes C^{n-2j}\varphi^+_j\\
&\qquad\qquad \bmod\ U_q^{+,0}(N^-_{>0})^2\otimes U_q^{+,0}N^+_{\ge 0}\quad (n>0),\\
\Delta x^-_{-n} &\equiv C^{-n}\otimes x^-_{-n}+\sum_{j=0}^{n}C^{-n+j}x^-_{-j}\otimes C^{-2j}\varphi^-_{-n+j}\\
&\qquad\qquad \bmod\ U_q^{-,0}(N^-_{\le 0})^2\otimes U_q^{-,0}N^+_{<0}\quad (n\ge 0),\\
\Delta b_n &\equiv b_n\otimes C^n+C^{2n}\otimes b_n\\
&\qquad\qquad \bmod\ U_q^{+,0}N^-_{>0}\otimes U_q^{+,0}N^+_{\ge 0}\quad (n>0),\\
\Delta b_{-n} &\equiv b_{-n}\otimes C^{-2n}+C^{-n}\otimes b_{-n}\\
&\qquad\qquad \bmod\ U_q^{-,0}N^-_{\le 0}\otimes U_q^{-,0}N^+_{<0}\quad (n>0).
\end{aligned}
$$

From these we find that

$$
\begin{aligned}
x^+_n(v'_i\otimes u)&=v'_i\otimes q^{2n+i}x^+_n u\quad (n\ge 0),\\
b_n(v'_i\otimes u)&=v'_i\otimes q^{2n}b_n u\quad (n>0),
\end{aligned}
$$

for $i=0,1$ and $u\in V(-\Lambda_j)$. On the other hand, we have for $m\ge 0$

$$
\begin{aligned}
\Bigl(\sum_{n\ge 0}x^+_n z^{-n-1}\Bigr)\overline{v}'_m &= \Bigl(\sum_{n\in\mathbb{Z}}x^+_n z^{-n-1}\Bigr)\overline{v}'_m\\
&= z^{-m-2}\exp\Bigl(\sum_{k>0}\frac{b_k}{k}(q^{-1}z)^{-k}\Bigr)\overline{v}'_{m+2}.
\end{aligned}
$$

Eq. (2.31) follows from these relations. Calculating similarly, we find

$$
\begin{aligned}
x^+_{-n}(v'_i\otimes\overline{v}'_m)&=(q^{2n-1}x^+_{-n}v'_i)\otimes\overline{v}'_m\quad (n>0),\\
b_{-n}(v'_i\otimes u)&=(q^{2n}b_{-n}v'_i)\otimes\overline{v}'_m\quad (n>0),
\end{aligned}
$$

and

$$
\Bigl(\sum_{n<0}x^+_n z^{-n-1}\Bigr)(v'_i\otimes\overline{v}'_m)=q^{2i+1}z^i\exp\Bigl(\sum_{k>0}\frac{b_{-k}}{k}z^k\Bigr)(v'_{i+2}\otimes\overline{v}'_m).
$$

Acting with

$$\Delta((x_0^-)^{(r)}) = \sum_{s=0}^{r} q^{s(r-s)}(x_0^-)^{(s)} \otimes (x_0^-)^{(r-s)} t_1^{-s},$$
$$\Delta((x_{-1}^+)^{(r)}) = \sum_{s=0}^{r} q^{s(r-s)} t_1^{-(r-s)}(x_{-1}^+)^{(s)} \otimes (x_{-1}^+)^{(r-s)} C^{-s},$$

we obtain

$$(x_0^-)^{(m+2+i)}(v_i' \otimes \overline{v}_{m+2}') = v_{-i}' \otimes (x_0)^{(m+2)}\overline{v}_{m+2}',$$
$$(x_{-1}^+)^{(m+2+i)}(v_{-i}' \otimes (x_0)^{(m+2)}\overline{v}_{m+2}') = (-1)^{m+i+1}q^{-2m-1}(v_{i+2}' \otimes \overline{v}_m'),$$

where we have used

$$(x_{-1}^+)^{(m+1)}(x_0^-)^{(m+2)}\overline{v}_{m+2}' = (-q)^{-m-1}\overline{v}_m',$$
$$(x_{-1}^+)^{(i+1)}v_{-i}' = (-q)^{-i}v_{i+2}',$$

which follow from Lemma 2.10. Combining these relations we arrive at (2.32). □

The proof of Proposition 2.4 is now complete.

2.7. **Intertwining operators.** In this section we show Proposition 2.5.

For integrable modules M, N, define the completion of $M \otimes N$ by $(M \otimes N)^\wedge = \sum_{\mu,\nu} \prod_{\xi \in Q_+} M_{\mu+\xi} \otimes N_{\nu-\xi}$, where $Q_+ = \mathbb{Z}_{\geq 0}\alpha_0 + \mathbb{Z}_{\geq 0}\alpha_1$. Similarly define $(M_1 \otimes \cdots \otimes M_p)^\wedge = ((M_1 \otimes \cdots \otimes M_{p-1})^\wedge \otimes M_p)^\wedge$. From the definition we have

$$(V_z^{\otimes n})^\wedge = V^{\otimes n} \otimes_K K[[z_1/z_2, \ldots, z_{n-1}/z_n]][z_1^{\pm 1}, \ldots, z_n^{\pm 1}].$$

Consider the intertwiner of U_q-modules of the form

$$\widetilde{\Psi}(z) : V(\Lambda_i) \longrightarrow \big(V_z \otimes V(\Lambda_{1-i})\big)^\wedge. \tag{2.33}$$

We write $\widetilde{\Psi}(z)u = \sum_{\epsilon,n} z^{-n} v_\epsilon \otimes \widetilde{\Psi}_{\epsilon,n} u$ for $u \in V(\Lambda_i)$, and set $\widetilde{\Psi}_\epsilon(z) = \sum_{n\in\mathbb{Z}} \widetilde{\Psi}_{\epsilon,n} z^{-n}$. We choose the normalization $\widetilde{\Psi}_{-,0}v_0 = v_1$, $\widetilde{\Psi}_{+,0}v_1 = v_0$. For more details concerning $\widetilde{\Psi}(z)$, we refer to [7], Chapter 6.

Denote by

$$\langle\ ,\ \rangle\ :\ V(\Lambda_i) \otimes V(-\Lambda_i) \to K \tag{2.34}$$

a U_q-linear mapping normalized as $\langle v_i, \overline{v}_{-i}\rangle = 1$.

Iterating (2.33), we obtain a U_q'-linear map

$$\psi_n : V(\Lambda_i) \otimes V(-\Lambda_j) \longrightarrow (V_z^{\otimes n})^\wedge \qquad (n \equiv i - j \bmod 2),$$
$$\psi_n(u \otimes v) = \rho_n^{(i,j)}(z_1, \ldots, z_n) \sum_{\epsilon_1,\ldots,\epsilon_n} \langle \widetilde{\Psi}_{\epsilon_n}(z_n) \ldots \widetilde{\Psi}_{\epsilon_1}(z_1)u, v\rangle v_{\epsilon_1} \otimes \cdots \otimes v_{\epsilon_n},$$

where

$$\rho_n^{(i,j)}(z_1,\dots,z_n) = (-q)^{-l(l-j)} \prod_{\substack{1\le k\le n\\ k:\mathrm{odd}}} z_k^{-(k-1)/2} \prod_{\substack{1\le k\le n\\ k:\mathrm{even}}} z_k^{-k/2+i} \times \prod_{1\le k<k'\le n} \frac{(q^2 z_k/z_{k'};q^4)_\infty}{(z_k/z_{k'};q^4)_\infty}. \tag{2.35}$$

Here we have set $n=2l+i-j$, and $(z;p)_\infty=\prod_{n\ge0}(1-p^n z)$ (note that $(z;q^4)_\infty^{\pm1}\in A[[z]]$). We have included the scalar factor $\rho_n^{(i,j)}(z_1,\dots,z_n)$, so that the normalization condition

$$\psi_n(v_i\otimes\overline{v}_{n-i}) = \overbrace{v_+\otimes\cdots\otimes v_+}^{n\ \text{times}} \tag{2.36}$$

holds. This has the effect of shifting degrees as

$$\deg(u\otimes v) = \deg\psi_n(u\otimes v) + \frac{(n-i)^2-j}{4}. \tag{2.37}$$

Proposition 2.5 is a consequence of (2.38) below. (Note that $\tilde\psi_n$ in Proposition 2.5 is the restriction of ψ_n to $F_n^{(i,j)}$.)

Proposition 2.12. *We have*

$$\psi_n(F_n^{(i,j)}) = V(n(\Lambda_1-\Lambda_0)), \tag{2.38}$$

$$\psi_n(F_{n+2}^{(i,j)}) = 0, \tag{2.39}$$

$$\psi_n\bigl(V_A(\Lambda_i)\otimes V_A(-\Lambda_j)\bigr) \subset (V_{A,z}^{\otimes n})^\wedge. \tag{2.40}$$

Proof. Since $V\bigl(n(\Lambda_0-\Lambda_1)\bigr)$ is generated by $v_+^{\otimes n}$, the assertion (2.38) is clear from (2.36). The assertion (2.39) follows from the fact that $\psi_n(v_0\otimes\overline{v}_{n+2})=0$, which is obvious from consideration of weights. To see (2.40), because of the cyclicity of $V_A(\Lambda_i)\otimes V_A(-\Lambda_j)$, it suffices to show that the vector $\psi_n(v_i\otimes\overline{v}_{-j})$ belongs to $(V_{A,z}^{\otimes n})^\wedge$. Set

$$\rho_n^{(i,j)}(z_1,\dots,z_n)\langle\widetilde\Psi_{\epsilon_n}(z_n)\cdots\widetilde\Psi_{\epsilon_1}(z_1)v_i,\overline{v}_{-j}\rangle = \frac{a_{\epsilon_1,\dots,\epsilon_n}(z_1,\dots,z_n)}{\prod_{1\le r<s\le n}(1-q^{-2}z_r/z_s)}. \tag{2.41}$$

As before, let $n=2l+i-j$. From [7], eq.(9.8) and p.116, we have

$$a_{\underbrace{-\cdots-}_{l}\underbrace{+\cdots+}_{n-l}}(z_1,\dots,z_n) = (-q)^{l(l-1)/2}\prod_{1\le r\le l} z_r^i \prod_{l+1\le r\le n} z_r^{-l} \times \prod_{1\le r<s\le l}(1-q^{-2}z_r/z_s)\prod_{l+1\le r<s\le n}(1-q^{-2}z_r/z_s), \tag{2.42}$$

$$a_{\dots,\pm,\mp,\dots}(\dots,z_k,z_{k+1},\dots) = \frac{(z_k/z_{k+1}-q^2)(z_k/z_{k+1})}{q(1-z_k/z_{k+1})}a_{\dots,\mp,\pm,\dots}(\dots,z_{k+1},z_k,\dots) - \frac{(1-q^2)(z_k/z_{k+1})^t}{q(1-z_k/z_{k+1})}a_{\dots,\mp,\pm,\dots}(\dots,z_k,z_{k+1},\dots). \tag{2.43}$$

Here $t = 0$ for the upper sign and 1 for the lower sign. In the right hand side, there is no pole at $1 - z_k/z_{k+1} = 0$. It follows that $a_{\epsilon_1,\dots,\epsilon_n}(z_1,\dots,z_n)$ are Laurent polynomials in $q, z_1, \dots, z_n$. Expanding the right hand side of (2.41) into a Laurent series in $z_1,\dots,z_n$, we see that all coefficients are Laurent polynomials in q. □

3. Functional model

3.1. **The space $\mathcal{F}_n$.** In this and the next subsections, we introduce various spaces of polynomials used to give a realization of the tensor product $V(\Lambda_i) \otimes V(-\Lambda_j)$. As before, we set $K = \mathbb{C}(q)$, $A = \mathbb{C}[q, q^{-1}]$.

For $0 \le l \le n$, let $\mathcal{F}_{n,l}$ be the space of polynomials $P(X_1,\dots,X_l)$ with coefficients in $K[z_1^{\pm 1},\dots,z_n^{\pm 1}]$, satisfying the following conditions:

$$P \text{ is symmetric in } X_1,\dots,X_l, \tag{3.1}$$

$$\deg_{X_i} P \le n-1, \tag{3.2}$$

$$P|_{X_1=q^{-2}X_2=z_k^{-1}} = 0 \quad \text{for } 1 \le k \le n \text{ and } l \ge 2. \tag{3.3}$$

We set

$$\mathcal{F}_{A,n,l} := \mathcal{F}_{n,l} \cap A[z_1^{\pm 1},\dots,z_n^{\pm 1}][X_1,\dots,X_l],$$
$$\mathcal{F}_n := \oplus_{l=0}^n \mathcal{F}_{n,l}, \quad \mathcal{F}_{A,n} := \oplus_{l=0}^n \mathcal{F}_{A,n,l}.$$

For each subset $M \subset \{1,\dots,n\}$ with $\sharp M = l$, define $w_M^{(n)}(X_1,\dots,X_l) \in \mathcal{F}_{A,n,l}$ by

$$w_M^{(n)}(X_1,\dots,X_l) := \mathrm{Sym}\Bigl(G_{m_1}^{(n)}(X_1)\cdots G_{m_l}^{(n)}(X_l) \prod_{1\le k<k'\le l} \frac{q^{-1}X_k - qX_{k'}}{X_k - X_{k'}}\Bigr). \tag{3.4}$$

Here $M = \{m_1,\dots,m_l\}$ $(1 \le m_1 < \cdots < m_l \le n)$, Sym stands for the symmetrization

$$(\mathrm{Sym}\, f)(X_1,\dots,X_l) := \sum_{\sigma\in\mathfrak{S}_l} f(X_{\sigma(1)},\dots,X_{\sigma(l)}),$$

and

$$G_m^{(n)}(X) := q^{m-n} \prod_{k=1}^{m-1}(1 - q^{-2}z_kX) \prod_{k=m+1}^{n}(1 - z_kX).$$

We will write (3.4) also as $w^{(n)}_{\epsilon_1,\dots,\epsilon_n}(X_1,\dots,X_l) = w^{(n)}_{\epsilon_1,\dots,\epsilon_n}(X_1,\dots,X_l|z_1,\dots,z_n)$, where $\epsilon_1,\dots,\epsilon_n \in \{+,-\}$ are related to M via $M = \{j \mid \epsilon_j = -\}$. The polynomials (3.4) arise naturally in the framework of the quantum inverse scattering method (see (C.1) and (C.2) in [8]). We have the transformation property

$$\begin{aligned} &w^{(n)}_{\dots,\epsilon'_{j+1},\epsilon'_j,\dots}(X_1,\dots,X_l|\dots,z_{j+1},z_j,\dots) \\ &= \sum_{\varepsilon_j,\varepsilon_{j+1}} w^{(n)}_{\dots,\epsilon_j,\epsilon_{j+1},\dots}(X_1,\dots,X_l|\dots,z_j,z_{j+1},\dots)\bigl(R(z_j/z_{j+1})^{-1}\bigr)_{\epsilon_j,\epsilon_{j+1};\epsilon'_j,\epsilon'_{j+1}}, \end{aligned} \tag{3.5}$$

where

$$R(z) = \begin{pmatrix} 1 & & & \\ & \dfrac{(1-z)q}{1-q^2z} & \dfrac{1-q^2}{1-q^2z} & \\ & \dfrac{(1-q^2)z}{1-q^2z} & \dfrac{(1-z)}{1-q^2z} & \\ & & & 1 \end{pmatrix}.$$

We assign a degree to an element $P \in \mathcal{F}_n$ by setting

$$\deg X_p = -1, \quad \deg z_j = 1. \tag{3.6}$$

Define a completion of $\widehat{\mathcal{F}}_n = \oplus_{l=0}^n \widehat{\mathcal{F}}_{n,l}$ by

$$\widehat{\mathcal{F}}_{n,l} := \mathcal{F}_{n,l} \bigotimes_{K[z_1^{\pm 1},\ldots,z_n^{\pm 1}]} K[[z_1/z_2,\ldots,z_{n-1}/z_n]][z_1^{\pm 1},\ldots,z_n^{\pm 1}].$$

Similarly, we define the completion $\widehat{\mathcal{F}}_{A,n}$.

We have

Lemma 3.1. [20] *For each l $(0 \le l \le n)$, the set of polynomials $w_M^{(n)}(X_1,\ldots,X_l)$ with $\#M = l$ constitutes a free basis of $\widehat{\mathcal{F}}_{A,n,l}$ over*

$$\widehat{A}_n = A[[z_1/z_2,\ldots,z_{n-1}/z_n]][z_1^{\pm 1},\ldots,z_n^{\pm 1}].$$

Proof. Let $\mathcal{P}_{n,l}$ be the set of partitions $\lambda = (\lambda_1,\ldots,\lambda_l)$ satisfying $\lambda_1 \le n-1$. Consider an element of $\widehat{A}_n[X_1,\ldots,X_l]$ of the form

$$P_{n,l} = \sum_{\lambda \in \mathcal{P}_{n,l}} c_\lambda m_\lambda, \tag{3.7}$$

where m_λ is the monomial symmetric polynomial corresponding to λ. The condition (3.3) is equivalent to a set of linear relations for the coefficients $c_\lambda \in \widehat{A}_n$. The linear relations are defined over $A[z_1^{\pm 1},\ldots,z_n^{\pm 1}]$.

Let $d_{n,l}$ be the dimension of the space of solutions $(c_\lambda)_{\lambda \in P_{n,l}}$ where c_λ belongs to the quotient field of $\widehat{A}_n$. If we specialize the relation (3.3) to $q = \sqrt{-1}$, it is equivalent to

$$P_{n,l}|_{X_1=-X_2} = 0. \tag{3.8}$$

This is because $P_{n,l}\big|_{X_1=X,X_2=-X}$ is a polynomial in X of degree at most $2(n-1)$ and has $2n$ zeroes $\pm z_k^{-1}$. Therefore, we have $d_{n,l} \le \binom{n}{l}$.

Note that $G_m^{(n)}(z_j^{-1}) = 0$ if $m < j$, and $G_m^{(n)}(z_m^{-1})$ is an invertible element in $\widehat{A}_n$.

For subsets $M = \{m_1 < \cdots < m_l\}$, $J = \{j_1 < \cdots < j_l\}$, we write $M \le J$ if and only if $m_a \le j_a$ for all a. Then $\le$ is a partial ordering. By induction on l one can show that

$$w_J^{(n)}(z_{m_1}^{-1},\ldots,z_{m_n}^{-1}) = 0$$

unless $M \le J$, and $w^{(n)}_J(z^{-1}_{j_1},\dots,z^{-1}_{j_n})$ is invertible in $\widehat{A}_n$. Using this triangularity one can show that if $\sum_{M,\sharp(M)=l} c_M w^{(n)}_M = 0$, then $c_M = 0$ for all M. Therefore, we have $d_{n,l} \ge \binom{n}{l}$, and $d_{n,l} = \binom{n}{l}$. In conclusion, we proved that $w^{(n)}_M$ such that $\sharp(M) = l$ constitute a basis of the vector space of solutions to (3.3) over the quotient field of $\widehat{A}_n$.

Let $f \in \mathcal{F}_{A,n,l}$ be written as

$$f = \sum_M c_M w^{(n)}_M.$$

We show that $c_M \in \widehat{A}_n$. Suppose that $J \subset \{1,\dots,n\}$ is a maximum element in $\{M; c_M \ne 0\}$. We have $f(z^{-1}_{j_1},\dots,z^{-1}_{j_l}) = c_J w^{(n)}_J(z^{-1}_{j_1},\dots,z^{-1}_{j_n})$, $f(z^{-1}_{j_1},\dots,z^{-1}_{j_l}) \in \widehat{A}_n$, and moreover, $w^{(n)}_J(z^{-1}_{j_1},\dots,z^{-1}_{j_n})$ is invertible in $\widehat{A}_n$. Therefore, we have $c_J \in \widehat{A}_n$. Applying the same argument to $f - c_J w^{(n)}_J \in \widehat{A}_n[X_1,\dots,X_l]$, and proceeding inductively, we see that $c_M \in \widehat{A}_n$ for all M. □

The following result can be extracted from [22] and Propositions C.1, C.2 in [8]. There the symmetric group $\mathfrak{S}_n$ acts on $\mathcal{F}_n$ by the permutation of variables $z_1,\dots,z_n$.

Proposition 3.2. (i) *The space $\mathcal{F}_n$ has a structure of a U_q-module of level* 0*. The action of U'_q commutes with the multiplication by elements of $K[z_1^{\pm1},\dots,z_n^{\pm1}]$, and $DP = q^d P$, where $d = \deg P$ is given by* (3.6)*, and t_1 acts on $\mathcal{F}_{n,l}$ by the multiplication of q^{n-2l}. The action preserves the $\mathfrak{S}_n$-invariant subspace*:

$$U_q \mathcal{F}_n^{\mathfrak{S}_n} \subset \mathcal{F}_n^{\mathfrak{S}_n}. \tag{3.9}$$

(ii) *There exists an injective morphism of U_q-modules given by*

$$\mathcal{C}_n : V_z^{\otimes n} \longrightarrow \mathcal{F}_n, \qquad v_{\epsilon_1}\otimes\cdots\otimes v_{\epsilon_n} \mapsto w^{(n)}_{\epsilon_1,\dots,\epsilon_n}(X_1,\dots,X_l). \tag{3.10}$$

Moreover $\mathcal{C}_n$ is $K[z_1^{\pm1},\dots,z_n^{\pm1}]$-linear.
(iii) *We have*

$$U_A \mathcal{F}_{A,n} \subset \mathcal{F}_{A,n}, \quad \mathcal{C}_n(V_{A,z}^{\otimes n}) \subset \mathcal{F}_{A,n}.$$

Explicit formulas for the action on $\mathcal{F}_n$ can be found in [8], Proposition C.1.

By Lemma 3.1, the morphism $\mathcal{C}_n$ can be extended to an isomorphism between the completions.

$$\widehat{\mathcal{C}}_n : (V_z^{\otimes n})\widehat{\ } \xrightarrow{\sim} \widehat{\mathcal{F}}_n,$$

Denote by $\varphi_n := \widehat{\mathcal{C}}_n \circ \psi_n$ the composition

$$\varphi_n \colon V(\Lambda_i)\otimes V(-\Lambda_j) \xrightarrow{\psi_n} (V_z^{\otimes n})\widehat{\ } \xrightarrow{\mathcal{C}_n} \widehat{\mathcal{F}}_n \qquad (n \equiv i-j \bmod 2).$$

Then φ_n is U'_q-linear. For a weight vector $u\otimes v \in V(\Lambda_i)\otimes V(-\Lambda_j)$, let $m = \langle h_1, \mathrm{wt}\,(u\otimes v)\rangle$. Then $\varphi_n(u\otimes v) \in \oplus_{l=0}^n \widehat{\mathcal{F}}_{n,l}$ has the only non-zero component for

$l=(n-m)/2$. We denote it by $P^{u\otimes v}_{n,l}=P^{u\otimes v}_{n,l}(X_1,\dots,X_l|z_1,\dots,z_n)$. Explicitly we have

$$
\begin{aligned}
(3.11)\quad P^{u\otimes v}_{n,l} &= \rho^{(i,j)}_n(z_1,\dots,z_n) \\
&\times \sum_{\epsilon_1,\dots,\epsilon_n}\langle\widetilde{\Psi}_{\epsilon_n}(z_n)\cdots\widetilde{\Psi}_{\epsilon_1}(z_1)u,v\rangle w^{(n)}_{\epsilon_1,\dots,\epsilon_n}(X_1,\dots,X_l|z_1,\dots,z_n).
\end{aligned}
$$

Note also that

$$
\deg(u\otimes v)=\frac{(n-i)^2-j}{4}+\deg P^{u\otimes v}_{n,l}.
$$

Let us consider the special case $v_i\otimes\overline{v}_{-j}$.

Proposition 3.3. *For $i,j\in\{0,1\}$ and $n\ge 0$ with $n=2l+i-j$, we have*

$$
\varphi_n(v_i\otimes\overline{v}_{-j})=(-)^{l(l-1)/2}q^\gamma\prod_{a=1}^{l}X_a^{1-j}\prod_{1\le a\ne b\le l}(X_a-q^{-2}X_b), \tag{3.12}
$$

where $\gamma=l(l-1)/2-2l(1-j)-l(n-1)$.

Proof. The right hand side of (3.12) belongs to $\mathcal{F}_n$. From Lemma 3.1, we can therefore write

$$
(-)^{l(l-1)/2}q^\gamma\prod_{a=1}^{l}X_a^{1-j}\prod_{1\le a\ne b\le l}(X_a-q^{-2}X_b)=\sum_{\epsilon_1,\dots,\epsilon_n}c_{\epsilon_1,\dots,\epsilon_n}w^{(n)}_{\epsilon_1,\dots,\epsilon_n}, \tag{3.13}
$$

where $c_{\epsilon_1,\dots,\epsilon_n}$ are some rational functions in $q,z_1,\dots,z_n$. We are to show the relation

$$
c_{\epsilon_1,\dots,\epsilon_n}=\rho^{(i,j)}_n(z_1,\dots,z_n)\langle\widetilde{\Psi}_{\epsilon_n}(z_n)\cdots\widetilde{\Psi}_{\epsilon_1}(z_1)v_i,\overline{v}_{-j}\rangle. \tag{3.14}
$$

Specializing $X_1=q^2z_1^{-1},\dots,X_l=q^2z_l^{-1}$ in (3.13), we find that only the term with $\epsilon_1=\dots=\epsilon_l=-$, $\epsilon_{l+1}=\dots=\epsilon_n=+$ contributes. Comparing the result with the matrix element (2.42), we obtain (3.14) for $\epsilon_1=\dots=\epsilon_l=-$. The general case follows from this, since both sides of (3.14) share the same transformation property under the exchange of (z_k,ϵ_k) and (z_{k+1},ϵ_{k+1}). □

In general, the image of φ_n has the following properties.

Proposition 3.4. (i)

$$
\varphi_n(V(\Lambda_i)\otimes V(-\Lambda_j))\subset\mathcal{F}_n^{\mathfrak{S}_n},\quad \varphi_n(V_A(\Lambda_i)\otimes V_A(-\Lambda_j))\subset\mathcal{F}_{A,n}^{\mathfrak{S}_n}.
$$

(ii) *We have*

$$
\varphi_n(F^{(i,j)}_{n+2})=0. \tag{3.15}
$$

The induced map $\overline{\varphi}_n:F^{(i,j)}_n/F^{(i,j)}_{n+2}\to\mathcal{F}_n^{\mathfrak{S}_n}$ is injective.
(iii) *Let $u\otimes v\in V(\Lambda_i)\otimes V(-\Lambda_j)$ and $P^{u\otimes v}_{n,l}$ be as above. We have*

$$
\begin{aligned}
(3.16)\quad & P^{u\otimes v}_{n+2,l+1}(X_1,\dots,X_l,z^{-1}|z_1,\dots,z_n,z,q^2z) \\
&=q^\nu z^{-n-1+i}\prod_{a=1}^{l}(1-q^{-2}zX_a)(1-q^2zX_a)\times P^{u\otimes v}_{n,l}(X_1,\dots,X_l|z_1,\dots,z_n).
\end{aligned}
$$

Here $n=2l+i-j$, $\nu=-5l+i+j-2$ *for* n *even, and* $\nu=-5l+i+j-1$ *otherwise.*

Proof. By Proposition 3.3, we have $\varphi_n(v_i\otimes\overline{v}_{-j})\in\mathcal{F}^{\mathfrak{S}_n}_{A,n}$. Since $V(\Lambda_i)\otimes V(-\Lambda_j)=U_q(v_i\otimes\overline{v}_{-j})$ and $V_A(\Lambda_i)\otimes V_A(-\Lambda_j)=U_A(v_i\otimes\overline{v}_{-j})$, assertion (i) follows from (3.9).

We have (3.15) by (2.39). Moreover the composition

$$V(n(\Lambda_1-\Lambda_0))\xrightarrow{\sim}F^{(i,j)}_n/F^{(i,j)}_{n+2}\xrightarrow{\overline{\varphi}_n}\mathcal{F}^{\mathfrak{S}_n}_n$$

coincides with $\mathcal{C}_n$ which is injective. Hence $\overline{\varphi}_n$ is also injective, and we have (ii).

Let us show (3.16). It is enough to verify it assuming that q is a complex number with $|q|<1$. In this case, the intertwiners satisfy the properties ([7], eq.(6.40), with $v\in V(\Lambda_j)$):

$$\begin{aligned}&\widetilde{\Psi}_\epsilon(z')\widetilde{\Psi}_\epsilon(z)v=O(1)\qquad(z'\to q^2z),\\&\frac{(q^2z/z';q^4)_\infty}{(q^4z/z';q^4)_\infty}\bigl(\widetilde{\Psi}_+(z')\widetilde{\Psi}_-(z)-q^{-1}\widetilde{\Psi}_-(z')\widetilde{\Psi}_+(z)\bigr)v\\&\qquad\qquad=(-q)^{-j}v+O\bigl((z'-q^2z)\bigr)\qquad(z'\to q^2z).\end{aligned}$$

It is easy to verify the relations

$$\begin{aligned}&w^{(n+2)}_{\epsilon_1,\dots,\epsilon_n,\epsilon,\epsilon}(X_1,\dots,X_l,z^{-1}|z_1,\dots,z_n,z,q^2z)=0,\\&-qw^{(n+2)}_{\epsilon_1,\dots,\epsilon_n,+-}(X_1,\dots,X_l,z^{-1}|z_1,\dots,z_n,z,q^2z)\\&\quad=w^{(n+2)}_{\epsilon_1,\dots,\epsilon_n,-+}(X_1,\dots,X_l,z^{-1}|z_1,\dots,z_n,z,q^2z)\\&\quad=q^{-l-1}(1-q^2)\prod_{k=1}^n(1-q^{-2}z_k/z)\prod_{a=1}^l(1-q^{-2}zX_a)(1-q^2zX_a)\\&\qquad\times w^{(n)}_{\epsilon_1,\dots,\epsilon_n}(X_1,\dots,X_l|z_1,\dots,z_n).\end{aligned}$$

Specializing (3.11) to $X_{l+1}=z^{-1},z_{n+1}=z,z_{n+2}=q^2z$, and using these relations along with (2.35), we obtain (3.16). ☐

3.2. **The spaces $\widehat{\mathcal{Z}}^{(i,j)}$ and W_n.** Proposition 3.4 motivates us to consider the subspace $\widehat{\mathcal{Z}}^{(i,j)}[m]\subset\prod\mathcal{F}^{\mathfrak{S}_n}_{n,l}$, consisting of all sequences of polynomials $(P_{n,l})_{\substack{n\geq0\\ n-2l=m}}$ such that

$$P_{n+2,l+1}\text{ and }P_{n,l}\text{ are related by (3.16).}\tag{3.17}$$

We set

$$\begin{aligned}\widehat{\mathcal{Z}}^{(i,j)}_A[m]&=\widehat{\mathcal{Z}}^{(i,j)}[m]\cap\prod_{\substack{n\geq0\\ n-2l=m}}\mathcal{F}^{\mathfrak{S}_n}_{A,n,l},\\\widehat{\mathcal{Z}}^{(i,j)}&=\bigoplus_{\substack{m\in\mathbb{Z}\\ m\equiv i-j\bmod 2}}\widehat{\mathcal{Z}}^{(i,j)}[m],\\\widehat{\mathcal{Z}}^{(i,j)}_A&=\bigoplus_{\substack{m\in\mathbb{Z}\\ m\equiv i-j\bmod 2}}\widehat{\mathcal{Z}}^{(i,j)}_A[m].\end{aligned}$$

The space $\widehat{\mathcal{Z}}^{(i,j)}$ is a U_q-module by the componentwise action.

Define also a filtration of $\widehat{\mathcal{Z}}^{(i,j)}$ by setting

$$\widehat{\mathcal{Z}}^{(i,j)}_n = \bigoplus_{\substack{m\in\mathbb{Z}\\ m\equiv i-j \bmod 2}} \widehat{\mathcal{Z}}^{(i,j)}_n[m], \tag{3.18}$$

$$\begin{aligned}\widehat{\mathcal{Z}}^{(i,j)}_n[m] = \{(P_{n',l})_{\substack{n'\geq 0\\ n'-2l=m}} \in \widehat{\mathcal{Z}}^{(i,j)}[m] \mid \\ P_{n',l}=0 \text{ for all } n',l \text{ such that } n'<n\}.\end{aligned} \tag{3.19}$$

From Proposition 3.4, we have a morphism of U'_q-modules

$$\varphi = \prod_n \varphi_n \ : \ V(\Lambda_i)\otimes V(-\Lambda_j) \longrightarrow \widehat{\mathcal{Z}}^{(i,j)},$$

which satisfies $\varphi\big(F^{(i,j)}_n\big) \subset \widehat{\mathcal{Z}}^{(i,j)}_n$. By Proposition 3.4 (ii) and Proposition 2.2, φ is injective. Our goal is to show that, after an appropriate completion of $V(\Lambda_i)\otimes V(-\Lambda_j)$, the mapping φ becomes an isomorphism (see Theorem 3.7 below).

For that purpose, let us introduce the following spaces. Let $W_{n,l}$ denote the subspace of $\mathcal{F}^{\mathfrak{S}_n}_{n,l}$ consisting of elements P such that

$$P|_{X_1={z_1}^{-1}=(q^{-2}z_2)^{-1}} = 0 \quad \text{for } n\geq 2 \text{ and } l\geq 1.$$

Set further

$$\begin{aligned} W_{A,n,l} &:= W_{n,l}\cap \mathcal{F}_{A,n,l},\\ W^{\geq 0}_{n,l} &:= W_{n,l}\cap K[z_1,\ldots,z_n][X_1,\ldots,X_l],\\ W^{\geq 0}_{A,n,l} &:= W_{n,l}\cap A[z_1,\ldots,z_n][X_1,\ldots,X_l]. \end{aligned}$$

We set $W_n := \oplus_{l=0}^n W_{n,l}$, and similarly for $W_{A,n}$, $W^{\geq 0}_n$, $W^{\geq 0}_{A,n}$. From the explicit action (see [8], Proposition C.1) we have $U_qW_n\subset W_n$, $U_AW_{A,n}\subset W_{A,n}$ and $U^{\geq 0}_q W^{\geq 0}_n \subset W^{\geq 0}_n$.

Consider the isomorphisms of U'_q-modules

$$V\big(n(\Lambda_1-\Lambda_0)\big) \xrightarrow{\ \sim\ } F^{(i,j)}_n/F^{(i,j)}_{n+2} \xrightarrow{\ \sim\ } \varphi(F^{(i,j)}_n)/\varphi(F^{(i,j)}_{n+2}). \tag{3.20}$$

The first map given by Proposition 2.5 shifts the degree by $+s$, and the second map φ by $-s$, where $s=((n-i)^2-j)/4$. Hence the composition is U_q-linear. By Proposition 3.4 (ii), there are also injective canonical maps

$$\varphi(F^{(i,j)}_n)/\varphi(F^{(i,j)}_{n+2}) \rightarrowtail \widehat{\mathcal{Z}}^{(i,j)}_n/\widehat{\mathcal{Z}}^{(i,j)}_{n+2} \rightarrowtail W_n. \tag{3.21}$$

The composition of (3.20) and (3.21)

$$V(n(\Lambda_1-\Lambda_0)) \longrightarrow W_n \tag{3.22}$$

coincides with the restriction to $V(n(\Lambda_1-\Lambda_0)) \simeq U_q v_+^{\otimes n} \subset V_z^{\otimes n}$ of the map $\mathcal{C}_n$ defined in (3.10).

Proposition 3.5. *Let 1_n be the unit of $\mathcal{F}_{n,0}$. We have*

$$W_n = U_q 1_n, \quad W^{\geq 0}_n = U^{\geq 0}_q 1_n.$$

We defer the proof of Proposition 3.5 to the next subsection.

Theorem 3.6. *The morphisms* (3.21)*,* (3.22) *are isomorphisms.*

Proof. The map (3.22) is injective because so is $\mathcal{C}_n$. Proposition 3.5 shows that it is also surjective. □

Define the completed tensor product with respect to the filtration $\{F_n^{(i,j)}\}$

$$V(\Lambda_i)\widehat{\underset{F}{\otimes}}V(-\Lambda_j)=\varprojlim V(\Lambda_i)\otimes V(-\Lambda_j)/F_n^{(i,j)}.$$

By Proposition 2.2, the map $V(\Lambda_i)\otimes V(-\Lambda_j)\to V(\Lambda_i)\widehat{\underset{F}{\otimes}}V(-\Lambda_j)$ is injective. Clearly $\varprojlim \widehat{\mathcal{Z}}^{(i,j)}/\widehat{\mathcal{Z}}_n^{(i,j)}=\widehat{\mathcal{Z}}^{(i,j)}$. Theorem 3.6 implies that

$$V(\Lambda_i)\widehat{\underset{F}{\otimes}}V(-\Lambda_j)\longrightarrow \widehat{\mathcal{Z}}^{(i,j)}$$

is an isomorphism. Hence we arrive at the following result, which provides a 'functional realization' of the (completed) tensor product of level 1 and level -1 integrable modules.

Theorem 3.7. *We have an isomorphism*

$$V(\Lambda_i)\widehat{\underset{F}{\otimes}}V(-\Lambda_j)\xrightarrow{\sim}\widehat{\mathcal{Z}}^{(i,j)}.$$

3.3. **Relation with ∞-cycles.** In [9], sequences similar to the elements of $\widehat{\mathcal{Z}}^{(i,j)}$ have been considered under the name '∞-cycles'. The latter are closely related to the specialization of the former at $q=\sqrt{-1}$. The aim of this subsection is to clarify the connection between these objects.

First we recall the definitions given in [8, 9] [4]. Let $\mathcal{F}^{\rm skew}_{\mathbb{C},n,l}$ be the space of polynomials $P(X_1,\dots,X_l)$ with coefficients in $\mathbb{C}[z_1^{\pm1},\dots,z_n^{\pm1}]$, satisfying the following conditions:

P is skew-symmetric in $X_1,\dots,X_l$ (it is an empty condition when $l=0,1$),
$\deg_{X_i}P\le n-1$.

We denote by $\mathcal{F}^{\rm skew,\mathfrak{S}_n}_{\mathbb{C},n,l}$ the $\mathfrak{S}_n$-invariant subspace of $\mathcal{F}^{\rm skew}_{\mathbb{C},n,l}$. Let $W^{\rm skew}_{\mathbb{C},n,l}$ denote the subspace of $\mathcal{F}^{\rm skew,\mathfrak{S}_n}_{\mathbb{C},n,l}$ consisting of elements P such that

$$P|_{X_1={z_1}^{-1}=-{z_2}^{-1}}=0\quad\text{for } n\ge 2 \text{ and } l\ge 1.$$

We set

$$W^{\rm skew\ge 0}_{\mathbb{C},n,l}:=W^{\rm skew}_{\mathbb{C},n,l}\cap\mathbb{C}[z_1,\dots,z_n][X_1,\dots,X_l].$$

[4]The spaces $\mathcal{F}^{\rm skew}_{\mathbb{C},n,l}$, $W^{\rm skew}_{\mathbb{C},n,l}$, $W^{\rm skew\ge0}_{\mathbb{C},n,l}$ and $\widehat{\mathcal{Z}}^{\rm skew(0,j)}_{\mathbb{C}}$ defined below are denoted in [8, 9] by $\mathcal{F}_{n,l}$, $\widehat{W}_{n,l}$, $W_{n,l}$ and $\widehat{\mathcal{Z}}^{(j)}$, respectively.

Let further $\widehat{\mathcal{Z}}^{\mathrm{skew}(i,j)}_{\mathbb{C}}[m]$ denote the space of sequences $(P_{n,l})_{n-2l=m} \in \prod_{\substack{n\geq 0\\ n-2l=m}} \mathcal{F}^{\mathrm{skew}}_{\mathbb{C},n,l}$, satisfying the conditions

$$\begin{aligned} &P_{n+2,l+1}(X_1,\ldots,X_l,z^{-1}|z_1,\ldots,z_n,z,-z) \\ &\quad = z^{-n-1+i}\prod_{a=1}^{l}(1-X_a^2z^2)\cdot P_{n,l}(X_1,\ldots,X_l|z_1,\ldots,z_n). \end{aligned}$$

We set $\mathcal{F}^{\mathrm{skew},\mathfrak{S}_n}_{\mathbb{C},n} = \bigoplus_{l=0}^{n} \mathcal{F}^{\mathrm{skew},\mathfrak{S}_n}_{\mathbb{C},n,l}$, $W^{\mathrm{skew}}_{\mathbb{C},n} = \bigoplus_{l=0}^{n} W^{\mathrm{skew}}_{\mathbb{C},n,l}$, $\widehat{\mathcal{Z}}^{\mathrm{skew}(i,j)}_{\mathbb{C}} = \bigoplus_{\substack{m\in\mathbb{Z}\\ m\equiv i-j \bmod 2}} \widehat{\mathcal{Z}}^{\mathrm{skew}(i,j)}_{\mathbb{C}}[m]$.

The spaces $\mathcal{F}^{\mathrm{skew},\mathfrak{S}_n}_{\mathbb{C},n,l}$, $W^{\mathrm{skew}}_{\mathbb{C},n,l}$, $\widehat{\mathcal{Z}}^{\mathrm{skew}(i,j)}_{\mathbb{C}}$ admit an action of $U_{\sqrt{-1}}$ (see [8, 9]). As noted in (3.8), we have an embedding of $U_{\sqrt{-1}}$-modules

$$\begin{array}{rcl} \iota : \left(\mathcal{F}^{\mathfrak{S}_n}_{A,n,l}\right)_{\sqrt{-1}} & \longrightarrow & \mathcal{F}^{\mathrm{skew},\mathfrak{S}_n}_{\mathbb{C},n,l}, \\ P & \mapsto & c_{n,l}P\cdot\prod_{j<j'}\frac{X_j-X_{j'}}{X_j+X_{j'}}, \end{array} \tag{3.23}$$

where $c_{n,l}\in\mathbb{C}\backslash\{0\}$. We can choose $c_{n,l}$ so that we have a map

$$\iota : (\widehat{\mathcal{Z}}^{(i,j)}_{A})_{\sqrt{-1}} \longrightarrow \widehat{\mathcal{Z}}^{\mathrm{skew}(i,j)}_{\mathbb{C}},$$

and that when composed with the morphism

$$V_{\sqrt{-1}}(\Lambda_i)\otimes V_{\sqrt{-1}}(-\Lambda_j) \longrightarrow (\widehat{\mathcal{Z}}^{(i,j)}_{A})_{\sqrt{-1}}$$

the following are valid.

$$\begin{aligned} v_0\otimes\overline{v}_0 &\mapsto \mathbf{1}^{(0,0)}_{\sqrt{-1}} = (1,X,X\wedge X^3,\ldots)\in\widehat{\mathcal{Z}}^{\mathrm{skew}(0,0)}_{\mathbb{C}}[0], \\ v_0\otimes\overline{v}_1 &\mapsto \mathbf{1}^{(0,1)}_{\sqrt{-1}} = (1,X^2,X^2\wedge X^4,\ldots)\in\widehat{\mathcal{Z}}^{\mathrm{skew}(0,1)}_{\mathbb{C}}[1], \\ v_1\otimes\overline{v}_0 &\mapsto \mathbf{1}^{(1,0)}_{\sqrt{-1}} = (1,X,X\wedge X^3,\ldots)\in\widehat{\mathcal{Z}}^{\mathrm{skew}(1,0)}_{\mathbb{C}}[1], \\ v_1\otimes\overline{v}_{-1} &\mapsto \mathbf{1}^{(1,1)}_{\sqrt{-1}} = (1,1,1\wedge X^2,\ldots)\in\widehat{\mathcal{Z}}^{\mathrm{skew}(1,1)}_{\mathbb{C}}[0]. \end{aligned}$$

Here we used the wedge product notation

$$\begin{aligned} P_1\wedge P_2 &:= \frac{1}{l_1!l_2!}\,\mathrm{Skew}\,P_1(X_1,\ldots,X_{l_1})P_2(X_{l_1+1},\ldots,X_{l_1+l_2}), \\ (\mathrm{Skew}\,f)(X_1,\ldots,X_l) &= \sum_{\sigma\in\mathfrak{S}_l}(\mathrm{sgn}\,\sigma)f(X_{\sigma(1)},\ldots,X_{\sigma(l)}). \end{aligned}$$

The following result was proved in [8, 9].

Proposition 3.8. *We have*

$$W^{\mathrm{skew}}_{\mathbb{C},n} = U_{\sqrt{-1}}1_n,\quad W^{\mathrm{skew}\geq 0}_{\mathbb{C},n} = U^{\geq 0}_{\sqrt{-1}}1_n.$$

Let us finish the proof of Proposition 3.5.

Proof of Proposition 3.5. In order to show the equality $U_q^{\geq 0}1_n = W_n^{\geq 0}$, it is enough to show the equality of their characters. We have an inclusion of the A-modules

$U_A^{\geq 0}1_n \subset W_{A,n}^{\geq 0}$. They are free A-modules because both of them are A-submodules of the free A-module $A[z_1,\dots,z_n][X_1,\dots,X_l]$. Using Proposition 3.8, we have

$$\begin{aligned}\mathrm{ch}\, W_{\mathbb{C},n}^{\mathrm{skew}\geq 0} &= \mathrm{ch}\, U_{\sqrt{-1}}^{\geq 0}1_n \leq \mathrm{ch}\,(U_A^{\geq 0}1_n)_{\sqrt{-1}}\\ &= \mathrm{ch}\, U_A^{\geq 0}1_n \leq \mathrm{ch}\, W_{A,n}^{\geq 0} = \mathrm{ch}\,(W_{A,n}^{\geq 0})_{\sqrt{-1}} \leq \mathrm{ch}\, W_{\mathbb{C},n}^{\mathrm{skew}\geq 0}.\end{aligned}$$

Here the first inequality follows from the surjective map $(U_A^{\geq 0}1_n)_{\sqrt{-1}} \twoheadrightarrow U_{\sqrt{-1}}^{\geq 0}1_n$, and the last inequality follows from the injective map $(W_{A,n}^{\geq 0})_{\sqrt{-1}} \rightarrowtail W_{\mathbb{C},n}^{\mathrm{skew}\geq 0}$ induced by (3.23). In particular, we have $\mathrm{ch}\, U_A^{\geq 0}1_n = \mathrm{ch}\, W_{A,n}^{\geq 0}$, which implies $U_q^{\geq 0}1_n = W_n^{\geq 0}$. Noting that $(z_1\cdots z_n)^{-L}1_n \in U_q 1_n$, we obtain

$$W_n = \cup_L (z_1\cdots z_n)^{-L} W_n^{\geq 0} = \cup_L (z_1\cdots z_n)^{-L} U_q^{\geq 0}1_n = U_q 1_n.$$

□

Arguing similarly as in the previous subsection, we obtain the following isomorphism conjectured in [9].

Theorem 3.9.

$$V_{\sqrt{-1}}(\Lambda_i)\widehat{\underset{F}{\otimes}} V_{\sqrt{-1}}(-\Lambda_j) \simeq \widehat{\mathcal{Z}}_{\mathbb{C}}^{\mathrm{skew}(i,j)} \tag{3.24}$$

3.4. **Characters.** In what follows we use the standard symbol $(v)_n := \prod_{j=1}^n (1-v^j)$. The character of $W_{\mathbb{C},n,l}^{\mathrm{skew}\geq 0}$ was computed by Nakayashiki [18]. From the results of the previous subsection, we conclude that $W_n^{\geq 0}$ has the same character:

Corollary 3.10. *We have*

$$\mathrm{ch}_{\,v,z} W_n^{\geq 0} = \mathrm{ch}_{\,v,z} U_q^{\geq 0} v_+^{\otimes n} = \sum_{l=0}^n \frac{z^{n-2l}}{(v)_l (v)_{n-l}}.$$

In particular, taking the sum over n we obtain the known character formula of the integrable $\widehat{\mathfrak{sl}}_2$-modules of level -1 [16]:

$$\mathrm{ch}_{\,v,z}\big(V(-\Lambda_j)\big) = \sum_{n\equiv j \bmod 2} v^{(n^2-j)/4} \frac{\mathrm{ch}_{\,v,z}\pi_1^{*n}}{(v)_n}, \tag{3.25}$$

where

$$\mathrm{ch}_{\,v,z}\pi_1^{*n} = \sum_{l=0}^n \frac{(v)_n}{(v)_l (v)_{n-l}} z^{n-2l}$$

is the graded character of the fusion product of n copies of 2-dimensional irreducible $\mathfrak{sl}_2$-modules, see, e.g., (2.11) in [6].

In this paper we considered the filtration of the tensor product of the level 1 and -1 modules. The graded space associated with the induced filtration (2.18) of $V(-\Lambda_j)$ has the character (3.25). It is known that similar filtrations of tensor products exist in a very general setting, see [1] and Section A.2. In general, however, the subspaces defining the filtration are not generated by tensor products of extremal

vectors. In the case of integrable $\widehat{\mathfrak{sl}}_2$-modules of level $-k$, the following fermionic formula is known (the formula (2.14) in [6] in the limit $N \to \infty$):

$$\mathrm{ch}_{\,v,z} V\bigl(-(k-j)\Lambda_0 - j\Lambda_1\bigr) = \sum_{n \equiv j \bmod 2} K^{(k)}_{j,(n,\underbrace{0,\ldots,0}_{k-1})}(v) \frac{ch_{v,z}\pi_1^{*n}}{(v)_n}.$$

Here $K^{(k)}_{j,(n,0,\ldots,0)}(v)$ denotes the level restricted Kostka polynomial for $\mathfrak{sl}_2$, see e.g. (2.9) in [6]. We conjecture that the right hand side gives the character of the associated graded space of the induced filtration mentioned above.

Appendix A. Crystal and global bases

A.1. **Summary of known facts.** In this subsection, we briefly summarize some of the basic definitions and results on crystal and global bases which are relevant to the main text. We also give proofs of Propositions 2.2 and 2.6 at the end. Since our application in this paper is limited to the case of $U_q(\widehat{\mathfrak{sl}}_2)$, we do not spare time to prepare the notation for the general case. However, most of the statements are valid for arbitrary quantum affine algebras under suitable modifications.

Let R be a subring of K. We use $R = A$, A_0 or A_∞, where A_0 (resp., A_∞) is the ring of rational functions in q which are regular at $q = 0$ (resp., $q = \infty$). Let V be a vector space over K. An R-submodule $L \subset V$ is called an R-lattice of V if L is R-free and $V = K \otimes_R L$.

Let V be an integrable U_q-module. In particular, V has a weight space decomposition, $V = \oplus_{\lambda\in P} V_\lambda$. The operators $\tilde{f}_i$ and $\tilde{e}_i$ are defined as usual (see (2.2.2) in [11]). A crystal base of a U_q-module V is a pair $(L(V), B(V))$ of an A_0-lattice $L(V)$ and a basis $B(V)$ of the $\mathbb{C}$-vector space $L(V)/qL(V)$ satisfying the following conditions:

$$\tilde{e}_i L(V) \subset L(V), \quad \tilde{f}_i L(V) \subset L(V) \text{ for any } i, \tag{A.1}$$

$$\tilde{e}_i B(V) \subset B(V) \sqcup \{0\}, \quad \tilde{f}_i B(V) \subset B(V) \sqcup \{0\}, \tag{A.2}$$

$$L(V) = \oplus_{\lambda\in P} L(V)_\lambda, \quad B(V) = \bigsqcup_{\lambda \in P} B(V) \cap (L(V)/qL(V))_\lambda, \tag{A.3}$$

$$b' = \tilde{f}_i b \text{ if and only if } b = \tilde{e}_i b' \text{ for all } b, b' \in B(V) \text{ and } i. \tag{A.4}$$

Let V_A be an A-lattice of V, L_0 an A_0-lattice, and L_∞ an A_∞-lattice. Set $E = L_0 \cap L_\infty \cap V_A$. The triplet (L_0, L_∞, V_A) is called *balanced* if the mapping of $\mathbb{C}$-vector spaces

$$E \longrightarrow L_0/qL_0 \tag{A.5}$$

is an isomorphism. We denote by G the inverse map of this isomorphism. Suppose that (L_0, B) is a crystal base of V. The basis $\{G(b) \mid b \in B\}$ of V is called a *global basis*. We have $V = \oplus_{b\in B} KG(b)$ and $V_A = \oplus_{b \in B} AG(b)$.

There is an involution of U_q called the bar involution:

$$\overline{q} = q^{-1}, \quad \overline{e}_i = e_i, \quad \overline{f}_i = f_i, \quad \overline{q^h} = q^{-h}. \tag{A.6}$$

Let V be a U_q-module. An involution $^-$ of V is called a bar involution if $\overline{av} = \overline{a}\overline{v}$ holds for $a \in U_q$, $v \in V$.

An extremal module $V(\lambda)$ $(\lambda \in P)$ admits a bar involution such that $\overline{u_\lambda} = u_\lambda$. We take $V_A(\lambda) = U_A u_\lambda$ as its A-lattice. There exists a crystal base $(L(\lambda), B(\lambda))$ of $V(\lambda)$ such that the triple $(L(\lambda), \overline{L}(\lambda), V_A(\lambda))$ is balanced. The construction is as follows.

First, suppose that $\lambda \in P_+$. We define

$$L(\lambda) = \sum_{m=0}^{\infty} \sum_{i_1,\dots,i_m \in I} A_0 \tilde{f}_{i_1} \cdots \tilde{f}_{i_m} u_\lambda, \tag{A.7}$$

$$B(\lambda) = \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_m} u_\lambda \in L(\lambda)/qL(\lambda) \mid m \in \mathbb{Z}_{\geq 0}, i_1, \dots, i_m \in I\} \backslash \{0\}. \tag{A.8}$$

Proposition A.1. [10] *For $\lambda \in P_+$, the pair $(L(\lambda), B(\lambda))$ is a crystal base of $V(\lambda)$. The triplet $(L(\lambda), \overline{L(\lambda)}, V_A(\lambda))$ is balanced.*

Similarly, we can construct the crystal and global bases for $V(\lambda)$ when $\lambda \in P_-$. By abuse of notation we use $u_\lambda \in B(\lambda)$.

If $\lambda, \mu \in P_+$, we have an embedding

$$V(\lambda + \mu) \simeq U_q(u_\lambda \otimes u_\mu) \subset V(\lambda) \otimes V(\mu) \tag{A.9}$$

such that $u_{\lambda+\mu} \mapsto u_\lambda \otimes u_\mu$. It induces an embedding of crystal $B(\lambda + \mu) \subset B(\lambda) \otimes B(\mu)$.

Now, consider the tensor product $V(\lambda) \otimes V(-\mu)$ where $\lambda, \mu \in P_+$. The vector $u_\lambda \otimes u_{-\mu}$ is a cyclic vector of $V(\lambda) \otimes V(-\mu)$. In fact, we have

$$U_A(u_\lambda \otimes u_{-\mu}) = V_A(\lambda) \otimes V_A(-\mu). \tag{A.10}$$

There exists a unique bar involution of $V(\lambda) \otimes V(-\mu)$ such that $\overline{u_\lambda \otimes u_{-\mu}} = u_\lambda \otimes u_{-\mu}$. In general, $\overline{u \otimes v}$ is not equal to $\overline{u} \otimes \overline{v}$. However, we have

$$\overline{u_\lambda \otimes v} = u_\lambda \otimes \overline{v}, \quad \overline{v \otimes u_{-\mu}} = \overline{v} \otimes u_{-\mu}. \tag{A.11}$$

Proposition A.2. [15] *The pair $(L(\lambda) \otimes L(-\mu), B(\lambda) \otimes B(-\mu))$ is a crystal base of $V(\lambda) \otimes V(-\mu)$, and the triplet $(L(\lambda) \otimes L(-\mu), \overline{L(\lambda) \otimes L(-\mu)}, V_A(\lambda) \otimes V_A(-\mu))$ is balanced.*

In general, it is not true that $G(b_1 \otimes b_2) = G(b_1) \otimes G(b_2)$ for $b_1 \in B(\lambda)$ and $b_2 \in B(-\mu)$. However, we have

$$G(u_\lambda \otimes b) = u_\lambda \otimes G(b) \text{ for any } b \in B(-\mu), \tag{A.12}$$

$$G(b \otimes u_{-\mu}) = G(b) \otimes u_{-\mu} \text{ for any } b \in B(\lambda). \tag{A.13}$$

Let $\lambda \in P$, and write it as $\lambda = \xi - \eta$ where $\xi, \eta \in P_+$. We have a surjection

$$p_{\xi,\eta} : V(\xi) \otimes V(-\eta) \to V(\lambda) \tag{A.14}$$

sending $u_\xi \otimes u_{-\eta}$ to u_λ. We set $L(\lambda) = p_{\xi,\eta}(L(\xi) \otimes L(-\eta))$. The map $p_{\xi,\eta}$ induces $\overline{p}_{\xi,\eta} : (L(\xi)/qL(\xi)) \otimes (L(-\eta)/qL(-\eta)) \to L(\lambda)/qL(\lambda)$. We set $B(\lambda) = \overline{p}_{\xi,\eta}(B(\xi) \otimes B(-\eta)) \backslash \{0\}$.

Proposition A.3. [15, 11] *The pair $(L(\lambda), B(\lambda))$ is a crystal base of $V(\lambda)$, and the triplet $(L(\lambda), \overline{L(\lambda)}, V_A(\lambda))$ is balanced. For $b \in B(\xi) \otimes B(-\eta)$ we have*

$$p_{\xi,\eta}(G(b)) = G(\overline{p}_{\xi,\eta}(b)). \tag{A.15}$$

The crystal base $(L(\lambda), B(\lambda))$ and the global base $G(b)$ ($b \in B(\lambda)$) are independent of the choice of $\xi, \eta \in P_+$ such that $\lambda = \xi - \eta$. In fact, they are obtained from a universal object called the modified quantized enveloping algebra. The modified quantized enveloping algebra $\tilde{U}_q$ is

$$\tilde{U}_q := \oplus_{\lambda \in P} U_q a_\lambda \text{ where } U_q a_\lambda = U_q / \sum_{h \in P^*} U_q(q^h - q^{\langle h, \lambda \rangle}). \tag{A.16}$$

For any $\xi, \eta \in P_+$ such that $\lambda = \xi - \eta$, we denote by $\Phi_{\xi,\eta} : U_q a_\lambda \to V(\xi) \otimes V(-\eta)$ the U_q-linear mapping which sends a_λ to $u_\xi \otimes u_{-\eta}$.

Proposition A.4. *There exists a unique A_0-lattice $\tilde{L}_\lambda$ of $U_q a_\lambda$ and a unique basis $\tilde{B}_\lambda$ of $\tilde{L}_\lambda / q\tilde{L}_\lambda$ ($\lambda \in P$) satisfying the following properties.*

(i) *The triplet $(\tilde{L}_\lambda, \overline{\tilde{L}_\lambda}, U_A a_\lambda)$ is balanced.*
(ii) *The image of $\tilde{L}_\lambda$ by $\Phi_{\xi,\eta}$ is equal to $L(\xi) \otimes L(-\eta)$.*
(iii) *Let $\overline{\Phi}_{\xi,\eta} : \tilde{L}_\lambda / q\tilde{L}_\lambda \to L(\xi) \otimes L(-\eta) / qL(\xi) \otimes L(-\eta)$ be the induced map. Then $\overline{\Phi}_{\xi,\eta}$ gives a bijection between $\{b \in \tilde{B}_\lambda \mid \overline{\Phi}_{\xi,\eta}(b) \neq 0\}$ and $B(\xi) \otimes B(-\eta)$. For $b \in \tilde{B}_\lambda$ we have*

$$\Phi_{\xi,\eta}(G(b)) = G(\overline{\Phi}_{\xi,\eta}(b)). \tag{A.17}$$

(iv) *The set $\tilde{B}_\lambda$ has a structure of crystal such that $B(\xi) \otimes B(-\eta) \sqcup \{0\} \subset \tilde{B}_\lambda \sqcup \{0\}$ is an embedding which commutes with the action of $\tilde{e}_i, \tilde{f}_i$ ($i \in I$).*
(v) *The set $\tilde{B}_\lambda$ is equal to the inductive limit $\varinjlim_{\xi,\eta \to \infty} B(\xi) \otimes B(-\eta) \sqcup \{0\}$, where we use the dominance ordering in P_+ in taking the limit.*

We write $\widetilde{B}_\lambda$ for $B(U_q a_\lambda)$.

Proposition A.5. *Let $\Phi_\lambda \colon U_q a_\lambda \to V(\lambda)$ be the surjective morphism sending a_λ to u_λ. Then the induced morphism $\overline{\Phi}_\lambda \colon \tilde{L}_\lambda / q\tilde{L}_\lambda \to L(\lambda)/qL(\lambda)$ satisfies $\overline{\Phi}_\lambda(\tilde{B}_\lambda) \subset B(\lambda) \sqcup \{0\}$. Moreover $\{b \in \tilde{B}_\lambda \,;\, \overline{\Phi}_\lambda(b) \neq 0\} \to B(\lambda)$ is bijective, and $\Phi_\lambda(G(b)) = G(\overline{\Phi}_\lambda(b))$.*

Let $\mu \in P$. Suppose that $-w\mu \in P_+$ for some $w \in W$. Namely, the weight μ is an extremal weight in the weight space of the lowest weight module $V(w\mu)$. We identify, $V(\mu) \simeq V(w\mu)$, $B(\mu) \simeq B(w\mu)$, and the extremal vector $S_{w^{-1}} u_{w\mu} \in V(w\mu)$ with $u_\mu \in V(\mu)$.

Proposition A.6. *There exists a subset B_μ^+ of $B(\mu)$ such that*

$$U_q^+ u_\mu = \oplus_{b \in B_\mu^+} K G(b). \tag{A.18}$$

Proposition A.7. *Suppose that $\mu = \xi - \eta$ where $\xi, \eta \in P_+$. There exists an embedding of crystals*

$$B(\mu) \subset B(\xi) \otimes B(-\eta) \tag{A.19}$$

such that

$$B^+_\mu = u_\xi \otimes B(-\eta). \tag{A.20}$$

Let $\lambda \in P_+$ and $\mu \in P$. By Proposition A.3, the tensor product $V(\lambda) \otimes V(\mu)$ has the global base $G(b)$ where $b \in B(\lambda) \otimes B(\mu)$. Consider the submodule $N_{\lambda,\mu} = U_q(u_\lambda \otimes u_\mu) \subset V(\lambda) \otimes V(\mu)$.

Proposition A.8. *There exists a subset $B_{\lambda,\mu}$ of $B(\lambda) \otimes B(\mu)$ such that*

$$N_{\lambda,\mu} = \oplus_{b \in B_{\lambda,\mu}} KG(b). \tag{A.21}$$

The set $B_{\lambda,\mu} \sqcup \{0\}$ is invariant by $\tilde{f}_i, \tilde{e}_i$.

This proposition follows from

Lemma A.9. [12] *Let M be an integrable module with a crystal base $(L(M), B(M))$ and a bar involution. Let M_A be an A-lattice of M such that $(L(M), \overline{L(M)}, M_A)$ is balanced. Let N^+ be a U^+-submodule of M such that*

$$N^+ = \oplus_{b \in B_{N^+}} KG(b) \text{ for a subset } B_{N^+} \subset B(M). \tag{A.22}$$

Set

$$B_N = \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_m} b \mid b \in B_{N^+}\} \backslash \{0\}. \tag{A.23}$$

Then, we have

$$U_q N^+ = \oplus_{b \in B_N} KG(b). \tag{A.24}$$

Here we give proofs of Propositions 2.2 and 2.6.

Proof of Proposition 2.2. We take $\lambda = \Lambda_i$ and $\mu = \mu_n = \mathrm{wt}\, \overline{v}_{-(n-i)}$ where $n \equiv i - j \bmod 2$ in Proposition A.8. Let us prove that

$$B_{\lambda,\mu} \cap (u_\lambda \otimes B(\mu)) = u_\lambda \otimes B^+_\mu. \tag{A.25}$$

We take ξ, η as in Proposition A.7. Then, we have $B(\mu) \subset B(\xi) \otimes B(-\eta)$. We can choose ξ, η in such a way that if $\langle h_i, \lambda \rangle = 0$ then $\langle h_i, \xi \rangle = 0$. Note that if $\langle h_i, \xi \rangle = 0$ we have $\tilde{f}_i u_\xi = 0$. Therefore, from the tensor product rule for $\tilde{f}_i$ (see [10], (2.4.3)), we have

$$\tilde{f}_i (u_\lambda \otimes u_\xi) = \begin{cases} \tilde{f}_i u_\lambda \otimes u_\xi & \text{if } \langle h_i, \lambda \rangle > 0; \\ 0 & \text{otherwise.} \end{cases} \tag{A.26}$$

By Lemma A.9, we know that $B_{\lambda,\mu}$ is obtained from $u_\lambda \otimes B^+_\mu$ by applying $\tilde{f}_i$'s. We have

$$B^+_\mu = u_\xi \otimes B(-\eta) \tag{A.27}$$

by (A.20). Since $u_\lambda \otimes u_\xi \in B(\lambda + \xi)$ and $B(\lambda + \xi) \subset B(\lambda) \otimes B(\mu)$ is invariant by $\tilde{f}_i$, we have $B_{\lambda,\mu} \subset B(\lambda + \xi) \otimes B(-\eta)$.

We have

$$B_{\lambda,\mu}\cap(u_\lambda\otimes B(\mu))\subset(B(\lambda+\xi)\cap u_\lambda\otimes B(\xi))\otimes B(-\eta). \tag{A.28}$$

From (A.26) it follows that $B(\lambda+\xi)\cap u_\lambda\otimes B(\xi)=u_\lambda\otimes u_\xi$. Using (A.27) we have (A.25).

Let us prove (2.2). It suffices to show that

$$\bigcap_n B_{\lambda,\mu_n}=\{0\}.$$

Suppose that $b_1\otimes b_2\in B_{\lambda,\mu_n}$. Let us show that the actions of $\tilde{e}_i$'s bring $b_1\otimes b_2$ to $u_\lambda\otimes b_2'\in B_{\lambda,\mu_n}$. If $b_1\neq u_\lambda$, we have an i such that $\tilde{e}_ib_1\neq 0$. By the tensor product rule, we see that there exists an $l>0$ such that

$$(\tilde{e}_i)^l(b_1\otimes b_2)=\tilde{e}_ib_1\otimes(\tilde{e}_i)^{l-1}b_2\neq 0.$$

Since $B(\lambda)$ is connected, the assertion follows from this.

Now, we have $u_\lambda\otimes b_2\in B_{\lambda,\mu_n}$ for all n. By (A.25) we have $b_2\in B_{\mu_n}$ for all n. Since $\cap_n B_{\mu_n}=\{0\}$, we have the assertion (2.2). □

Proof of Proposition 2.6. We prove the first isomorphism. The second isomorphism then follows. It suffices to prove

$$U_A(v_i\otimes\overline{v}_{n-i})\cap U_q(v_i\otimes\overline{v}_{n+2-i})=U_A(v_i\otimes\overline{v}_{n+2-i}).$$

Using the notation in the proof of Proposition 2.2, we have

$$\begin{aligned}U_A(v_i\otimes\overline{v}_{n-i})&=\oplus_{b\in B_{\lambda,\mu_n}}AG(b),\\ U_q(v_i\otimes\overline{v}_{n+2-i})&=\oplus_{b\in B_{\lambda,\mu_{n+2}}}KG(b),\\ U_A(v_i\otimes\overline{v}_{n+2-i})&=\oplus_{b\in B_{\lambda,\mu_{n+2}}}AG(b).\end{aligned}$$

The assertion is clear from these equalities. □

A.2. **Filtration on $V(\xi)\otimes V(-\eta)$.** In the previous subsection, we prepared basic definitions and results which are used in the main text in the construction of the filtration of the tensor product of level 1 and level -1 $U_q(\widehat{\mathfrak{sl}}_2)$-modules. The submodules which constitute the filtration in this case are generated by single vectors of the form $v_i\otimes\overline{v}_{n-i}$, and these vectors $v_n,\overline{v}_n$ are the extremal vectors. In a more general situation, i.e., $V(\xi)\otimes V(\eta)$ where $\xi,\eta\in P$ for affine quantum algebras other than $U_q(\widehat{\mathfrak{sl}}_2)$ and/or level k of $\xi,-\eta$ is greater than 1, the existence of similar filtrations was proved in [1]. In the below, we briefly state the construction.

Let $B(\infty)$ be the crystal of U_q^- [10]. It has a unique element u_∞ which has weight zero, and is given in the form

$$B(\infty)=\{\tilde{f}_{i_1}\cdots\tilde{f}_{i_n}u_\infty\}\backslash\{0\}.$$

For $b\in B(\infty)$ we have $\varepsilon_i(b)=\max\{n\mid\tilde{e}_i^nb\neq 0\}\geq 0$ and $\langle h_i,\mathrm{wt}\,b\rangle+\varepsilon_i(b)=\varphi_i(b)$. Note that $\varphi_i(b)$ is finite and can be negative. Similarly, we have the crystal $B(-\infty)$ of U_q^+:

$$B(-\infty)=\{\tilde{e}_{i_1}\cdots\tilde{e}_{i_n}u_{-\infty}\}\backslash\{0\}.$$

The weight of $u_{-\infty}$ is zero and $\varphi_i(b) = \max\{n \mid \tilde{f}_i^n b \neq 0\} \geq 0$.

Let $\lambda \in P$. We denote by T_λ the crystal consisting of a single element t_λ such that $\varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty$. Suppose that b is an element of a crystal such that $\varphi_i(b)$ is finite. The equality $\varepsilon_i(t_\lambda) = -\infty$ implies that $x(b \otimes t_\lambda) = xb \otimes t_\lambda$ for $x = \tilde{f}_i$ or $\tilde{e}_i$. Similarly, if $\varepsilon_i(b)$ is finite we have $x(t_\lambda \otimes b) = t_\lambda \otimes xb$.

We have an isomorphism of crystals

$$B(U_q a_\lambda) \simeq B(\infty) \otimes T_\lambda \otimes B(-\infty).$$

The right hand side has a decomposition into the crystals $B(\zeta)$ ($\zeta \in P$). The decomposition is appropriately described by introducing the star crystal structure on $B(\tilde{U}_q) = \sqcup_{\lambda \in P} B(U_q a_\lambda)$.

Let $*$ be an anti-involution of U_q given by

$$q^* = q, \quad e_i^* = e_i, \quad f_i^* = f_i, \quad (q^h)^* = q^{-h}.$$

We define $*$ on $\tilde{U}_q$ by $a_\lambda^* = a_{-\lambda}$. It induces involutions of $B(\infty)$, $B(-\infty)$ and $B(\tilde{U}_q)$, which we also denote by $*$. We have

$$(b_1 \otimes t_\lambda \otimes b_2)^* = b_1^* \otimes t_{-\lambda - \mathrm{wt}\, b_1 - \mathrm{wt}\, b_2} \otimes b_2^*.$$

We define crystal structures on $B(\infty)$, $B(-\infty)$ and $B(\tilde{U}_q)$ by

$$\tilde{e}_i^* = * \circ \tilde{e}_i \circ *, \quad \tilde{f}_i^* = * \circ \tilde{f}_i \circ *.$$

We call the actions of $\tilde{e}_i^*$ and $\tilde{f}_i^*$ the star crystal actions. They commute with the original crystal actions $\tilde{e}_j$ and $\tilde{f}_j$. We have the structure of bi-crystal on $B(\infty)$, $B(-\infty)$ and $B(\tilde{U}_q)$. We have $\mathrm{wt}^*(b) = \mathrm{wt}\,(b^*)$. In particular, $\mathrm{wt}^*(U_q a_\lambda) = -\lambda$. or equivalently, $\mathrm{wt}^*(b_1 \otimes t_\lambda \otimes b_2) = -\lambda$. See [13] for other useful formulas of the star crystal actions on $B(\tilde{U}_q)$.

Let $\mathfrak{g} = \mathfrak{sl}_2$, and denote the simple root by $\alpha = 2\varpi$. It is easy to see that we have decompositions

$$B(\infty) \otimes T_{n\varpi} \otimes B(-\infty) \\ = \begin{cases} B(n\varpi) \sqcup B((n+2)\varpi) \sqcup B((n+4)\varpi) \sqcup \ldots & \text{if } n \geq 0; \\ B(n\varpi) \sqcup B((n-2)\varpi) \sqcup B((n-4)\varpi) \sqcup \ldots & \text{if } n < 0, \end{cases}$$

as crystals, forgetting the star crystal actions. For example, $\{u_\infty \otimes t_0 \otimes u_{-\infty}\}$ is isomorphic to $B(0)$ in the crystal action. In fact, it is also isomorphic to $B(0)$ in the star crystal action. Therefore, the subset $\{u_\infty \otimes t_0 \otimes u_{-\infty}\}$ of $B(\infty) \otimes T_0 \otimes B(-\infty)$ is isomorphic to $B(0) \times B(0)$ as bi-crystal. For bi-crystals, we use the symbol $\times$. It is not a product of crystals. The crystal structure of the second component represents the star crystal structure.

Similarly, $B(\varpi)\times B(-\varpi)$ is contained in the union $B(\infty)\otimes T_\varpi\otimes B(-\infty)\cup B(\infty)\otimes T_{-\varpi}\otimes B(-\infty)$. The identification is

$$\begin{aligned}(u_\varpi,u_{-\varpi}) &= u_\infty\otimes t_\varpi\otimes u_{-\varpi},\\ (u_{-\varpi},u_{-\varpi}) &= \tilde{f}u_\infty\otimes t_\varpi\otimes u_{-\varpi},\\ (u_{-\varpi},u_{\varpi}) &= u_\infty\otimes t_{-\varpi}\otimes u_{-\varpi},\\ (u_\varpi,u_\varpi) &= u_\infty\otimes t_\varpi\otimes \tilde{e}u_{-\varpi}.\end{aligned}$$

In general, we have the decomposition in the $\mathfrak{sl}_2$ case.

$$B(\tilde{U}_q(\mathfrak{sl}_2))=\sqcup_{n=0}^\infty B(n\varpi)\times B(-n\varpi),$$

where we have the identification $u_\infty\otimes t_{n\varpi}\otimes u_{-\infty}=(u_{n\varpi},u_{-n\varpi})$.

The decomposition of the bi-crystal $B(\tilde{U}_q(\mathfrak{g}))$ in the general case of quantum affine algebras was conjectured in [13], and proved in [1]. We denote by $B_0(\lambda)$ the connected component of $B(\lambda)$ which contains u_λ. There is an action of W on $\bigsqcup_{\lambda\in P}B(\lambda)\times B_0(-\lambda)$ induced by the isomorphism (2.6).

Proposition A.10. *We have*

$$B(\tilde{U}_q)=\sqcup_{\lambda\in P}(B(\lambda)\times B_0(-\lambda))/W. \tag{A.29}$$

In particular, we have $u_\infty\otimes t_\lambda\otimes u_{-\infty}=(u_\lambda,u_{-\lambda})$.

Now, we consider $\xi=\sum_i\xi_i\Lambda_i$ and $\eta=\sum_i\eta_i\Lambda_i\in P_+$ such that $\langle c,\xi\rangle=\langle c,\eta\rangle>0$. We set $\zeta=\xi-\eta$. The level of ζ is zero. The crystal $B(\xi)\otimes B(-\eta)$ is regarded as a subset of

$$B(U_qa_\zeta)=\sqcup_{\lambda\in P}(B(\lambda)\otimes B_0(-\lambda)_{-\zeta})/W. \tag{A.30}$$

We have the characterization of $B(\xi)\otimes B(-\eta)$ by using the star crystal structure.

Proposition A.11. *We have*

$$B(\xi)\otimes B(-\eta)=\{b\in B(U_qa_\zeta)\mid \varepsilon_i^*(b)\le\xi_i,\varphi_i^*(b)\le\eta_i \textit{ for all } i\in I\}. \tag{A.31}$$

Proof. By Proposition 2.1, we have

$$V(\xi)\otimes V(-\eta)\simeq U_qa_\zeta\Big/\left(\sum_i U_qf^{1+\xi_i}a_\zeta+\sum_i U_qe^{1+\eta_i}a_\zeta\right).$$

Set $I_{\xi,\eta}=\sum_i U_qf^{1+\xi_i}+\sum_i U_qe^{1+\eta_i}$. We have

$$I_{\xi,\eta}a_\zeta=U_qa_\zeta\cap\left(\sum_i\tilde{U}_qf^{1+\xi_i}+\sum_i\tilde{U}_qe^{1+\eta_i}\right).$$

Therefore, we have

$$(I_{\xi,\eta}a_\zeta)^*=a_{-\zeta}U_q\cap\left(\sum_i f_i^{1+\xi_i}\tilde{U}_q+\sum_i e_i^{1+\eta_i}\tilde{U}_q\right).$$

From this follows

$$I_{\xi,\eta}a_\zeta=\oplus_{b\in B_{\xi,\eta}}KG(b),$$

where

$$B_{\xi,\eta} = \left\{ b \in B(\tilde{U}_q)\,;\ \begin{array}{l}\text{there exists an } i \in I \text{ such that } \varepsilon_i(b^*) \geq 1 + \xi_i \text{ or} \\ \varphi_i(b^*) \geq 1 + \eta_i \text{ holds}\end{array} \right\}$$

Since $\varepsilon_i^*(b) = \varepsilon_i(b^*)$ and $\varphi_i^*(b) = \varphi_i(b^*)$, the assertion follows. □

Since $\langle h_i, \mathrm{wt}^*(b)\rangle + \varepsilon_i^*(b) = \varphi_i^*(b)$, the condition $\varepsilon_i(b^*) \leq \xi_i$ is equivalent to $\varphi_i(b^*) \leq \eta_i$.

Let $\lambda, \lambda' \in P$ be of level zero, we consider a partial ordering $\lambda \geq \lambda'$ if and only if $\mathrm{cl}(\lambda - \lambda') \in \sum_{i \in I\setminus\{0\}} \mathbb{Z}_{\geq 0}\alpha_i$. Here $\mathrm{cl}(\lambda) \in P/\mathbb{Z}\delta$ is the classical part of λ. We fix a total ordering for $\lambda, \lambda' \in P$ which is a refinement of the partial ordering. We denote it also by $\geq$.

We fix a set of representatives $P_+^{(0)} \subset \{\lambda \in P \mid \langle c, \lambda\rangle = 0\}$ with respect to the action of the Weyl group W in such a way that for any $\lambda \in P_0^+$, we have $\mathrm{cl}(\lambda) \in \sum_{i\in I\setminus\{0\}} \mathbb{Q}_{\geq 0}\alpha_i$. We denote the isotropy subgroup of λ by W_λ. Define a filtration $F_{\xi,\eta}^{\lambda}$ $(\lambda \in P_+^{(0)})$ of $V(\xi) \otimes V(-\eta)$ by

$$F_{\xi,\eta}^{\geq\lambda} = \bigoplus_{\lambda' \geq \lambda, \lambda' \in P_+^{(0)}} \left(\bigoplus_{b \in (B(\lambda') \times B_0(-\lambda')/W_{\lambda'}) \cap (B(\xi) \otimes B(-\eta))} KG(b) \right) \tag{A.32}$$

Similarly, we define $F_{\xi,\eta}^{>\lambda}$.

The following proposition follows from [1].

Proposition A.12. *The subspace $F_{\xi,\eta}^{\geq\lambda}$ is U_q-invariant. The subquotient $F_{\xi,\eta}^{\geq\lambda}/F_{\xi,\eta}^{>\lambda}$ is isomorphic to a direct sum of copies of $V(\lambda)$.*

For $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$, we have an explicit description of $B_0(-\lambda)_{-\zeta}$ by using "paths". See [17]. Note that the isotropy subgroup W_λ is trivial in this case. It is enough to consider the following two cases.

Case 1: $\lambda = 0$ and $\zeta = 0$.

Case 2: $\lambda = n(\Lambda_1 - \Lambda_0) + m\delta$ $(n > 0)$, $|\xi_1 - \eta_1| \leq n$ and $\xi_1 - \eta_1 \equiv n \bmod 2$.

Case 1 is trivial. In *Case* 2, we embed

$$B_0(-\lambda) \simeq B_0(n(\Lambda_1 - \Lambda_0)) \otimes T_{-m\delta} \subset (B(\Lambda_1 - \Lambda_0)^{\otimes n}) \otimes T_{-m\delta}.$$

We have the identification

$$B(\Lambda_1 - \Lambda_0) = \{z^\mu v_\varepsilon \mid \mu \in \mathbb{Z}, \varepsilon = \pm 1\}.$$

In this identification, an element $b = z^{\mu_1}v_{\varepsilon_1} \otimes \cdots \otimes z^{\mu_n}v_{\varepsilon_n} \otimes t_{-m\delta}$ belongs to $B_0(-\lambda)_{-\zeta}$ if and only if

$$\mu_{l+1} - \mu_l = \begin{cases} 0 & \text{if } (\varepsilon_l, \varepsilon_{l+1}) = (+,+), (-,+), (-,-); \\ 1 & \text{if } (\varepsilon_l, \varepsilon_{l+1}) = (+,-). \end{cases} \tag{A.33}$$

$$\mu_1 + \cdots + \mu_n = m, \tag{A.34}$$

$$\varepsilon_1 + \cdots + \varepsilon_n = \eta_1 - \xi_1. \tag{A.35}$$

The conditions $\varepsilon_i^*(b) \le \xi_i$ $(i = 0, 1)$ are equivalent to the "level restriction"

$$\max\{\varepsilon_1 + \cdots + \varepsilon_l; 1 \le l \le n\} \le \xi_0, \tag{A.36}$$

$$\min\{\varepsilon_1 + \cdots + \varepsilon_l; 1 \le l \le n\} \ge \xi_1. \tag{A.37}$$

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